Stellar Interiors

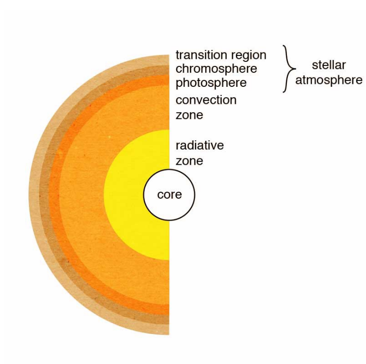

Fig. 1 The basic structure of a star.

Figure shows the basic structure of a star. (It is not scale!) The light we see comes from photons in the stellar atmosphere.

Hydrostatic Equilibrium

Let's imagine a small cylindrical volume of gas that is located in a larger region. There will be three forces acting on it: F_{grav} is the force of gravity, and two forces from the pressure, one from the pressure above: $P+\Delta P$ and one from pressure below: P. The net force from the pressure will be :

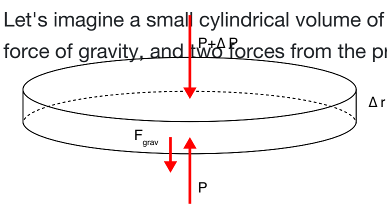

Fig. 2 A small volume of gas has three force vectors acting in the vertical directions: gravity, pressure from below and pressure from above.

$$F_{\text{pres}} = A[P-(P+\Delta P)] \quad (1)$$

where A is the area (whose surface normal points in the y direction) of the gas volume. The gravitation force will be given by:

$$F_{\text{grav}} = -\frac{GM_r\rho A\Delta r}{r^2} \quad (2)$$

where M_r is the mass enclosed by a sphere of radius r centered about the earth's center.

If this volume of gas is in equilibrium, then the pressure forces and the gravitational forces will be equal:

$$A[P-(P+\Delta P)] - \frac{GM_r \rho A \Delta r}{r^2} = 0 \quad (3)$$

Solving for ΔP:

$$\Delta P = -\frac{GM_r \rho}{r^2}\Delta r \quad (4)$$

If we consider this cylinder to be infinitely thin: $\Delta r \rightarrow= 0$, then we have a differential form:

$$\frac{dP}{dr} = -\frac{GM_r \rho}{r^2} \quad (5)$$

Eq. (5) is called the **equation of hydrostatic equilibrium** (for a spherical symmetric body).

To apply this to a star, we should acknowledge the limiting assumptions we made:

1. The star is spherical
2. The star is not rotating
3. The star is not expanding or contracting
4. Gravity and Pressure gradients are the only forces acting.

Pressure-radius

A better way to write this would be to include the dependencies:

$$\boxed{\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}} \quad (6)$$

We can see that there are 3 quantities that depend on r: Pressure (P), Mass (M) and Density (ρ). This makes a unique solution not possible. We need more equations!

Mass continuity

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho(r)} \quad (7)$$

The **equation of mass continuity** relates density and radial position and total mass. In short it says that the total mass is just the sum of all the infinitesimal spherical shells (i.e. onion skins)

Equation of State

$$PV = Nk_BT \quad (8)$$

The **equation of state** (8) relates density, temperature, and pressure for a gas. For many stars, the ideal gas law will work just fine.

Mean Molecular Weight

$$\mu \equiv \frac{\bar{m}}{m_H}$$

where $m_H = 1.6735... \times 10^{-27}$ kg, and $\bar{m}$ is the average mass of a gas particle in kg. μ therefore gives the the average mass of a free particles in the gas, in units of hydrogen. And, let's say that $n = \rho/\bar{m}$.

This allows us to write the ideal gas law as:

$$P_g = \frac{\rho k T}{\mu m_H} \quad (9)$$

The pressure integral

$$P = \frac{1}{3}\int_0^\infty n_p p v dp \quad (10)$$

Derivation of the pressure integral: Assuming a particle in the x direction, normal to a surface:

$$\mathbf{f}\Delta t = -\Delta p = 2p_x\,\hat{\mathbf{i}}$$

Since it must travel the distance Δx twice in the time Δt:

$$\Delta t = 2\frac{\Delta x}{v_x}$$

Thus the average force:

$$f = \frac{2p_x}{\Delta t} = \frac{p_x v_x}{\Delta x}$$

We can sub: $\frac{1}{3}pv \approx p_x v_x$

$$f(p) = \frac{1}{3}\frac{pv}{\Delta x}$$

The number of particles with a given momentum between p and $p + dp$ is $N_p dp$, then the total:

$$N = \int_0^\infty N_p dp$$

Contribution to the total force by all particles within that momentum range:

$$dF(p) = f(p)N_p dp = \frac{1}{3}\frac{N_p}{\Delta x}pv\,dp$$

Divide by A

$$n_p dp \equiv \frac{N_p}{\Delta V}dp$$

$$P = \frac{1}{3}\int_0^\infty n_p pv\,dp \quad (11)$$

For non-relativistic particles, $p = mv$:

$$P = \frac{1}{3}\int_0^\infty m n_v v^2 dv \quad (12)$$

Recall the Maxwell-Boltzmann distribution:

$$n_v dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-\frac{mv^2}{2kT}} dv$$

Putting this into Eq. (12) we'll get a simple expression:

$$P_g = nkT \quad (13)$$

(which is just the ideal gas law since $n = N/V$)

Stellar Energy Sources

Where does it come from?

The first suggestion to explain where stars got the energy was to look at the gravitational potential. Let's say we have two particles, then the gravitational potential energy between them is:

$$U = -G\frac{Mm}{r} \quad (14)$$

If the distance r between these particles decreases, then the potential energy must get more negative, and we would expect to find energy in other forms, kinetic for example.

How much energy is there in gravity?

Of course there are more than two particles in a star, so we'll have to expand that equation a bit. If a point mass dm is located outside a spherically symmetric mass M_r, then the force on that point mass will be directed to the center and will have a magnitude:

$$dF_{g,i} = G\frac{M_r dm_i}{r^2}$$

The potential energy of the point mass is thus:

$$dU_{g,i} = -G\frac{M_r dm_i}{r} \quad (15)$$

Now we'll consider a shell of thickness dr and mass dm:

$$dm = 4\pi r^2 \rho dr$$

where ρ is the density and $4\pi r^2 dr$ is the volume of the shell. Thus, eq (15) becomes

$$dU_g = -G\frac{M_r 4\pi r^2 \rho}{r}dr$$

We can integrate over the radius of the star R like this:

$$U_g = -4\pi G\int_0^R M_r \rho r \; dr \quad (16)$$

Of course we probably don't know ρ very well at this point, but we can guess based on it's average value:

$$\rho \sim \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$$

with M being the total mass of the star. Also:

$$M_r \sim \frac{4}{3}\pi r^3 \bar{\rho}$$

Putting this into eq. (16), we get:

$$U_g \sim \frac{16\pi^2}{15}G\bar{\rho}^2 R^5 \sim -\frac{3}{5}\frac{GM^2}{R} \quad (17)$$

Applying the virial theorum, we have:

$$E \sim -\frac{3}{10}\frac{GM^2}{R} \quad (18)$$

If we plug some sun values into this:

$$\frac{3}{10}\frac{GM_\odot^2}{R_\odot} \approx 1.1 \times 10^{41} \text{ J}$$

Now if the sun luminosity was constant at today's value: 3.84×10^{26} W, then this would mean there is enough energy to last for about 1×10^7 years (about 10 million years). This is notably too short given other observations from geology that suggest the earth and moon and things have been around for billions of years. There must be another source of energy contributing to the star's luminosity.

Intro to Nuclear Reactions

Fission

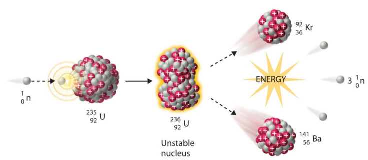

Fig. 3

http://www.nuclear-power.net/nuclear-power/fission/

When an atom breaks apart. Power plants...

Fusion

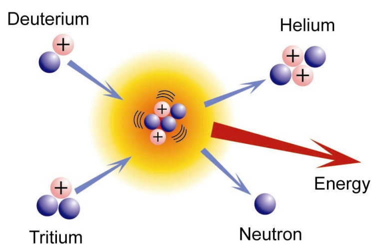

Fig. 4

http://www.nuclear-power.net/nuclear-power/nuclear-fusion/

When you smash two atoms together. Suns...

Binding Energies

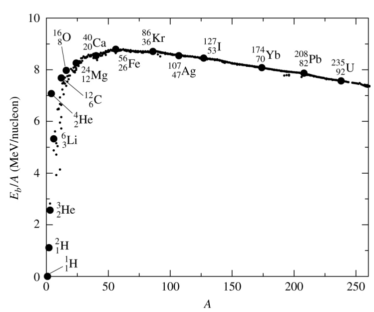

Fig. 5

Carroll & Ostlie, figure 10-9.

Take four Hydrogen atoms (1 proton & 1 electron), each have a mass of 1.00782503214 u. Together, they therefore have a combined mass of 4.03130013 u. One Helium-4 atom has a mass of 4.002603 u. This is obtained by adding the mass of the neutrons and the protons together. Thus, let's say we were able to mash 4 Hydrogen atoms together and make a Helium. Now there would be a change in mass between the initial and final constitutions: $\Delta m = 4.03130013 - 4.002603 = 0.028697\ \mathrm{u}$.

Since $E = mc^2$, we can see that this amount of mass is associated with an energy of 26.731 MeV. This is essentially the amount of energy in the formation of a Helium nucleus, aka the **binding energy**. (This is also the amount of energy that would be required to take apart a Helium nucleus.) The trick is to figure out exactly how we can combine 4 Hydrogen atoms to make a Helium. (that's the study of High energy physics earlier last century)

Bringing them together

One impediment is the Coulomb repulsion between two protons, both of which are positively charged.

As the separation distance between two charged (alike) particles decreases, the force of repulsion increases. The Coulomb potential between two protons located 1 fm apart

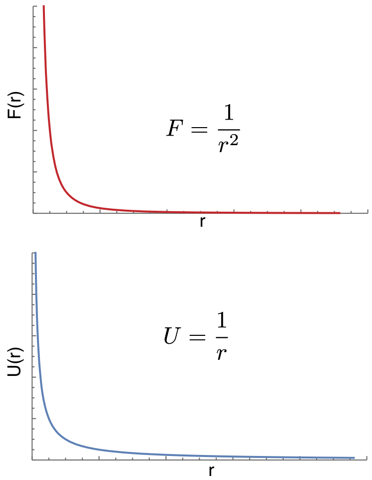

Fig. 6 The plots of the Coulomb force (top) and the Coulomb potential (bottom).

would be:

$$\approx \frac{(1.60\times10^{-19}\ \mathrm{C})^2}{4\pi(8.8\times10^{-12}\ \mathrm{C^2\ J^{-1}\ m^{-1}})(1.60\times10^{-19}\ \mathrm{J\ eV^{-1}})(10^{-15}\ \mathrm{m})}$$
$$\approx 1.4\times10^6\ \mathrm{eV}$$
$$\approx 1.4\ \mathrm{MeV}$$

If two protons wanted to get closer than 1 fm, they would need to have a kinetic of energy greater than this value to overcome the potential barrier. The energy of a proton at the center of the sun is about:

$$\langle E\rangle = \frac{3kT_c}{2} \approx \frac{3(1.38\times10^{-23}\ \mathrm{J\ K^{-1}})(1.47\times10^{7}\ \mathrm{K})}{2(1.6\times10^{-19}\ \mathrm{J\ eV^{-1}})} \approx 2\ \mathrm{keV} \quad (19)$$

This would be insufficient energy to get over the barrier. This is where quantum tunneling comes in.

Tunneling

Fig. 7 A 1-d simulation of a wave packet incident on a potential barrier. Some of the wave *tunnels* through the barrier.

Simulation based on code from https://github.com/jakevdp/pySchrodinger

Criteria for tunneling: a) the distance between protons should be comparable to the de Broglie wavelength:

$$\lambda_{DB} = \frac{h}{p} = 7\times10^{-13}\ \mathrm{m} \quad (20)$$

and b) the kinetic energy (E) of the proton needs to be comparable to the electrostatic potential energy at that separation.

$$U = \frac{e^2}{4\pi\epsilon_0}\frac{1}{\lambda_{DB}} = \frac{e^2}{4\pi\epsilon_0}\frac{(2m_pE)^{1/2}}{h} \approx E \quad (21)$$

Solving for the kinetic energy:

$$E \approx \left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{2m_p}{h^2} \approx \frac{1}{2\pi^2}\alpha^2 m_p c^2 \approx 3keV \quad (22)$$

Since this energy is comparable to the kinetic energy of a proton in the core of the sun, we can expect tunneling through the Coulomb barrier to be possible. The next step is to figure out just *how likely* such events are.

Energy / Tunneling

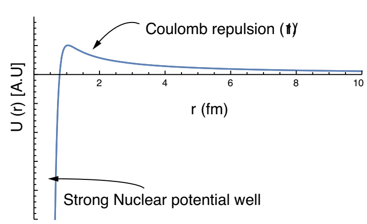

Fig. 8 A qualitative graph showing the what happens when two protons get close engouh for the strong force to dominate.

Cross Section

The **cross section** gives a probability of interaction. It is the number of reactions per target nucleus per time divided by the flux of incident particles.

$$\sigma(E) \equiv \frac{\text{number of reactions/nucleus/time}}{\text{number of incident particles/area/time}}$$

Cross Section

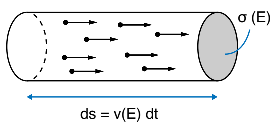

Fig. 9 Imagine a cylinder of length $ds = v(E)dt$ and area $\sigma(E)$. All the particles within the volume of the cylinder will reach the wall in the time dt.

If a particle hits the wall, then there is a reaction. This is an abstraction of the nuclear collision process. (there is no cylinder, no wall...)

n : just a particle

n_i : an incident particle

$n_{iE}dE$: # of incident particles having energies between E and $E + dE$ per unit volume

dN_E : # of particles that "hit the wall" / time or # of reactions / time

$n_E dE$: # of particles with an energy between E and $E + dE$, (might not be incident)

We have a target particle (x) and an incident particle (i). If the incident particle strikes the area $\sigma(E)$, then there will be a nuclear reaction.

Take the number of incident particles per unit volume having energy between E and $E + dE$ to be: $n_{iE}dE$. (This is like a density) To find the number of such particles inside the cylinder:

$$dN_E = \underbrace{n_{iE}dE}_{\text{density}} \underbrace{\sigma(E)v(E)dt}_{\text{volume}}$$

The number of particles with the appropriate kinetic energy is some fraction of the whole:

$$n_{iE}dE = \frac{\int_0^\infty n_{iE}dE}{\int_0^\infty n_E dE} n_E dE = \frac{n_i}{n} n_E dE$$

The number of reactions per target nucleus per time dt having energies between E and $E + dE$ is

$$\frac{\text{reactions per nucleus}}{\text{time interval}} = \frac{dN_E}{dt} = \sigma(E)v(E)\frac{n_i}{n}n_E dE$$

Lastly, if there are n_x targets per unit volume, then the total number of reactions per unit volume per unit time, given all possible energies is:

$$r_{ix} = \int_0^\infty n_x n_i \sigma(E) v(E) \frac{n_E}{n} dE \quad (23)$$

But, what exactly is σE?

$\sigma(E)$

$$\sigma(E) \propto \pi\lambda_{DB}^2 \propto \pi\left(\frac{h}{p}\right)^2 \propto \frac{1}{E} \quad (24)$$

(Since $K = E = \mu_m v^2/2 = p^2/2\mu_m$)

Also, for a barrier U and particle with energy E, the probability for tunneling through will go as:

$$\sigma(E) \propto e^{-2\pi^2 U/E} \quad (25)$$

In this case the U is the Coulomb potential: $U_c = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r}$ and E is just the kinetic energy: $E = \mu_m v^2/2$. We can assume the distance r is basically the de Broglie wavelength: $r \sim \lambda = h/p$.

In which case we obtain:

$$\sigma(E) \propto e^{-bE^{-1/2}} \quad (26)$$

where b is given by

$$b \equiv \frac{\pi \mu_m^{1/2} Z_1 Z_2 e^2}{2^{1/2} \epsilon_0 h}$$

We can combine eq (24) and (26) and define a general function $S(E)$ to get a functional form for $\sigma(E)$:

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}} \quad (27)$$

Finally, we can write a reaction rate as:

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/kT} dE \quad (28)$$

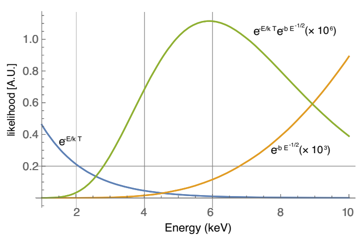

Fig. 10 The likelihood that a nuclear reaction will occur.

The peak will be at

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} \quad (29)$$

p-p chain

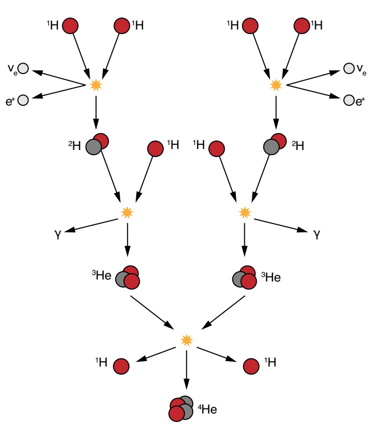

Fig. 11 The full p-p chain

$$4\,{}^1_1\mathrm{H} \rightarrow {}^4_2\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma$$

One way to convert Hydrogen to Helium is through the proton-proton chain (PP1). The overall reaction combines 4 protons to create 1 Helium-4 plus some other stuff.

The first step in this reaction take two protons and produces a deuterium, a positron, and an electron neutrino.

$${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{H} + e^+ + \nu_e \quad (30)$$

Then the deuterium combines with another proton to make a Helium-3 atom plus a gamma ray.

$${}^2_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma \quad (31)$$

Lastly, two Helium-3s combine to create Helium-4 and 2 protons

$${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He} + 2\,{}^1_1\mathrm{H} \quad (32)$$

CNO Cycle

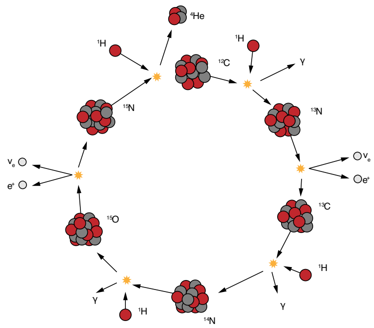

Fig. 12 The CNO cycle

In stars that more than $1.3M_\odot$, a different nuclear fusion process will dominate. This is the **CNO cycle**. It's more complicated, but the basic idea is that Carbon, Nitrogen, and Oxygen act a catalysts to speed the fusion of hydrogen. The net result is:

$$4p \rightarrow {}^{4}\mathrm{HE} + 2e^{+} + 2\nu_e + 3\gamma \quad (33)$$

Luminosity Gradient

$$dL = \epsilon\, dm \quad (34)$$

Here, ϵ is the total energy released per kilogram per second by all the star stuff: gravity and nuclear reactions together.

For spherically symmetric star, the mass of a thin shell of thicknessdr will be:

$$dm = dM_r = \rho dV = 4\pi r^2 \rho dr \quad (35)$$

which leads to:

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon \quad (36)$$

Energy Transport

Radiative

Radiation Pressure

Energy density of Blackbody radiation:

$$u_v dv = \frac{4\pi}{c} B_v dv = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT} - 1} \quad (37)$$

We can integrate this over all wavelengths/frequencies

$$u = \int_0^\infty u_\lambda d\lambda = \int_0^\infty u_\nu d\nu$$

Thus we obtain:

$$u = \frac{4\pi}{c} \int_0^\infty B_\lambda(T) d\lambda = \frac{4\sigma T^4}{c} = aT^4 \quad (38)$$

(a is the radiation constant given by: $7.565767 \times 10^{-16}\ \mathrm{J\ m^{-3}\ K^{-4}}$)

The **radiation pressure** can be shown to be one-third of the energy density:

$$P_{\mathrm{rad}} = \frac{aT^4}{3} \quad (39)$$

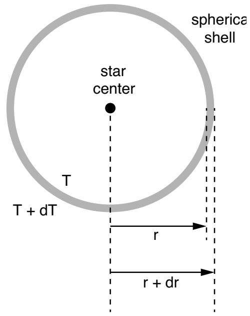

Fig. 13 A thin spherical shell centered on the star center. Let $dr \ll r$

The radiation pressure on the inner surface:

$$P_{\text{rad}}(r) = \frac{aT^4}{3} \quad (40)$$

The radiation pressure on the outer surface:

$$P_{\text{rad}}(r+dr) = \frac{a(T+dT)^4}{3} = \frac{a}{3}T^4\left(1+\frac{dT}{T}\right)^4 \approx \frac{a}{3}T^4\left(1+4\frac{dT}{T}\right) \quad (41)$$

(assuming $dt/T \ll 1$)

Thus, the net radiation force will be given by:

$$F_{\text{rad}} = [P_{\text{rad}}(r) - P_{\text{rad}}(r+dr)]4\pi r^2 \quad (42)$$

which using eqs. (40) and (41), we can get:

$$F_{\text{rad}} \approx -\frac{a}{3}4T^4\frac{dT}{T}4\pi r^2 = -\frac{16\pi}{3}ar^2T^3dT \quad (43)$$

The optical depth is given by:

$$d\tau = -\rho(r)\kappa(r)dr \quad (44)$$

where κ is the opacity. The probability that a photon will be absorbed while passing through the shell is$dP \approx d\tau$.

The rate at which photons carry energy through the shell is the luminosity, $L(r)$. Momentum is carried through the shell is just $L(r)/c$, since $p = E/c$.

The force on the shell, i.e. the rate at which photon momentum is transferred to the shell will be

$$F_{\text{rad}} = \frac{L(r)}{c}d\tau = -\frac{L(r)}{c}\rho(r)\kappa(r)dr \quad (45)$$

Combing eq. (45) equation with eq. (43), we get:

$$-\frac{16\pi}{3}ar^2T(r)^3dT = -\frac{L(r)}{c}\rho(r)\kappa(r)dr \quad (46)$$

This can be rearranged to yield the **equation of radiative energy transport**

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)L(r)}{16\pi ac\,T(r)^3r^2} \quad (47)$$

This equation related the luminosity to the temperature gradiant of a star.

Convective Transport

This is the **equation of convective energy transport**

$$\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right)\frac{T(r)}{P(r)}\frac{dP}{dr} \quad (48)$$

It shows that the temperature gradient is proportional to the pressure gradient. (γ is the **adiabatic index** and depends on the gas in question.)

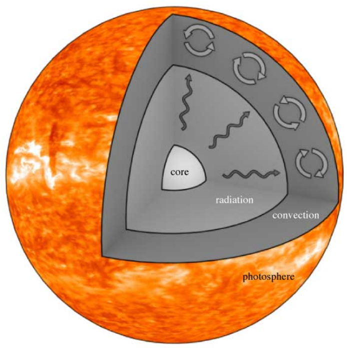

Fig. 14 The different transport modes are separated into regions.

Stellar Model Building

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dL}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)L(r)}{16\pi ac\, T(r)^3 r^2} \quad \text{[for radiative]}$$

$$= \left(1 - \frac{1}{\gamma}\right)\frac{T(r)}{P(r)}\frac{dP}{dr} \quad \text{[for convective]}$$

These are the basic equations that show the relationship between the main attributes of a star. They are 4 differential equations that relate: pressure, mass, density, temperature, energy generation, luminosity, and the radius of the star.

Boundary Conditions

In order to solve these equations, we need boundary conditions: At the center we should expect:

$$\left.\begin{array}{ll} M_r & \to 0 \\ L_r & \to 0 \end{array}\right\} \text{as } r \to 0$$

and at the surface of the star:

$$\left.\begin{array}{ll} T & \to 0 \\ P & \to 0 \\ \rho & \to 0 \end{array}\right\} \text{as } r \to R_*$$

1. Build a star mesa http://mesa-web.asu.edu/index.html