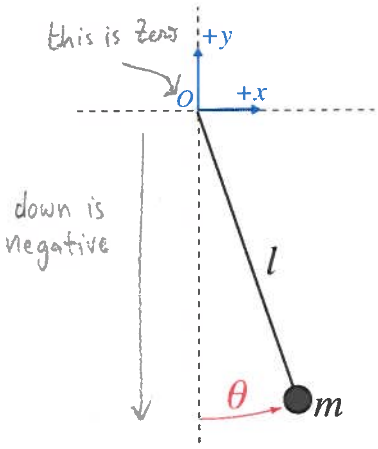
CCNY PHYS 35100 - Fall 2024 - Quiz 3 Name Solutions

Oct 28, 2024

Instructions: There is 1 question, with 9 short parts. Please show your work with as much detail as you can. Each part is worth the same amount. No calculators or notes.



1. In the figure is a pendulum near the surface of Earth.

a. Using the coordinates in the figure, write down the cartesian positions $\mathbf{r} = (x, y)$ of the mass in terms of θ and length of the string, l.

$$\vec{r} = (l\sin\theta, -l\cos\theta)$$

at $\theta = 0$, $x = 0$ and $y = -l$ ✓

b. Take the time derivatives of these positions to obtain the components of the cartesian velocity: $\mathbf{v} = (\dot{x}, \dot{y})$

$$\vec{v} = (l\dot{\theta}\cos\theta, l\dot{\theta}\sin\theta)$$

CHAIN RULE

θ is a function of time

c. Use your results from a) and b) to express the Lagrangian, $\mathcal{L}$, of this system.

$$\mathcal{L} = T - U = \frac{1}{2}mv^2 - mgy$$

just the y component from a)

$$= \frac{1}{2}m(l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2(\theta)) + mgl\cos\theta$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

d. Evaluate the Euler-Lagrange equation with your Lagrangian, $\mathcal{L}$, and obtain the expected equation of motion for the simple pendulum.

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

[θ is the only coordinate]

$$-mgl\sin\theta - ml^2\frac{d}{dt}(\dot{\theta}) = 0$$

$$-mgl\sin\theta - ml^2\ddot{\theta} = 0$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

negative sign is crucial

e. Now, imagine the the entire pendulum is hanging from an elevator ceiling. The elevator is accelerated upward at a constant rate, a. What is the position of the elevator in Cartsian Coords as a function of time? Write the new $\mathbf{r}_E = (x, y)$ components.

$r_E = (0, \frac{1}{2}at^2)$

position after time t for a constant acceleration

f. Similarly, write the velocity of the elevator in component form: $\mathbf{v}_E = (\dot{x}, \dot{y})$

$V_E = (0, at)$

velocity after time t for a constant acceleration

g. Now, add $\mathbf{r} + \mathbf{r}_E$ and $\mathbf{v} + \mathbf{v}_E$ and write out the Lagrangian for the mass accounting for all its motion. (You can assume it's not going high enough in altitude to significantly change the value of little g)

$\vec{r} + \vec{r}_E = (l\sin\theta, -l\cos\theta) + (0, \frac{1}{2}at^2)$

$= (l\sin\theta, \frac{1}{2}at^2 - l\cos\theta)$

$\vec{V} + \vec{V}_E = (l\dot{\theta}\cos\theta, l\dot{\theta}\sin\theta) + (0, at)$

$= (l\dot{\theta}\cos\theta, l\dot{\theta}\sin\theta + at)$

$\mathcal{L} = T - \mathcal{U}$

$T = \frac{1}{2}mv^2$ / $\mathcal{U} = mgy = mg(\frac{1}{2}at^2 - l\cos\theta)$

$\mathcal{L} = \frac{1}{2}m[l^2\dot{\theta}^2 + 2l\dot{\theta}\sin\theta\, at + a^2t^2] - mg(\frac{1}{2}at^2 - l\cos\theta)$

h. Use this new Lagrangian and the Euler-Langrange equation to find the equation of motion for the pendulum in the vertically accelerating elevator.

$\mathcal{L} = T - \mathcal{U}$

$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{2}m[2l\dot{\theta}\cos\theta\, at] - mgl\sin\theta = mlat\dot{\theta}\cos\theta - mgl\sin\theta$

$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{2}m[l^2 2\dot{\theta} + 2l\sin\theta\, at] = ml^2\dot{\theta} + mlat\sin\theta$

$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + mlat\dot{\theta}\cos\theta + mla\sin\theta$

$\cancel{mlat\dot{\theta}\cos\theta} - mgl\sin\theta = ml^2\ddot{\theta} + \cancel{mlat\dot{\theta}\cos\theta} + mla\sin\theta$

$\therefore ml^2\ddot{\theta} = -mgl\sin\theta - mla\sin\theta = -(g+a)ml\sin\theta$

$\rightarrow l\ddot{\theta} = -(g+a)\sin\theta$

negative sign!

i. What is the new frequency of oscillation for the pendulum in the accelerating elevator?

$\omega^2 = \frac{g+a}{l}$ $\therefore$

$\omega = \sqrt{\frac{g+a}{l}}$

if $a = 0$, reverts to $\sqrt{\frac{g}{l}}$ ✓