CCNY PHYS 35100 - Fall 2024 - Quiz 2 Name Solutions

Sep 30, 2024

Instructions: There are ~~3~~ 2 questions. Please show your work with as much detail as you can. Each part is worth the same amount. No calculators or notes.

1. A simple harmonic oscillator mass on a spring is described by the $x(t)$ function:

$$x(t) = A\cos(\omega t - \phi) \tag{1.1}$$

a. What are the velocity $v(t)$, and acceleration $a(t)$ functions based on this position function?

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t - \varphi)$$

$$a = \frac{dv}{dt} = -A\omega^2\cos(\omega t - \varphi)$$

b. Find an expression for the maximum velocity of this oscillator, in terms of the constants in (1.1).

v_{max} will be when $\sin(\omega t - \varphi) = 1$ $\therefore$ $|v_{max}| = A\omega$

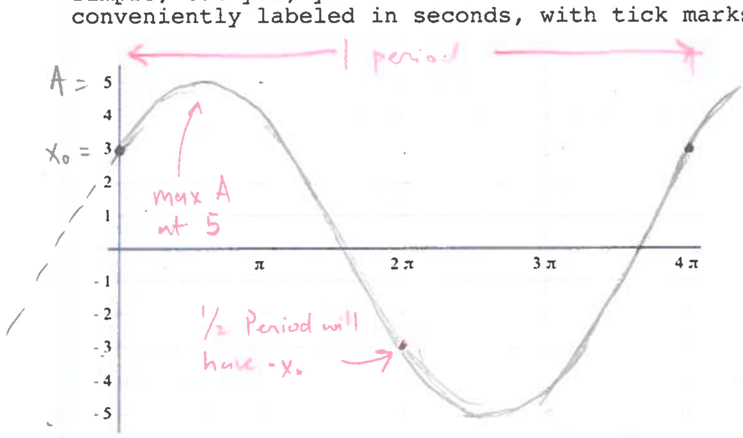
c. Using the functions from part a) obtain an expression that will give you the value of the amplitude of motion, A, in terms of the initial conditions: x_0 and v_0.

$$\left.\begin{aligned} x_0 &= A\cos(\varphi) \\ v_0 &= -A\omega\sin(\varphi) \end{aligned}\right\} \text{ use } \cos^2\varphi + \sin^2\varphi = 1$$

$$\left.\begin{aligned} x_0^2 &= A^2\cos^2(\varphi) \\ \frac{v_0^2}{\omega^2} &= A^2\sin^2\varphi \end{aligned}\right\rangle x_0^2 + \frac{v_0^2}{\omega^2} = A^2\cos^2\varphi + A^2\sin^2\varphi = A^2(\cos^2\varphi + \sin^2\varphi)$$

$$\therefore A = \underbrace{\left(x_0^2 + \frac{v_0^2}{\omega^2}\right)^{1/2}}_{\text{no } \varphi \text{ in here}}$$

d. At $t = 0$ the mass was located at 3 m from the origin and had a (negative) positive velocity of 2 m/s. We also know the mass is 8 kg and the spring constant is 2 N/m. Sketch a quantitatively correct plot showing 1 cycle of the $x(t)$ with the correct numerical values using the axis below. (don't worry, all the numbers were chosen so that the calculations will be very simple, but yes, you'll need to do some basic arithmetic) The horizontal axis is conveniently labeled in seconds, with tick marks at intervals of π.

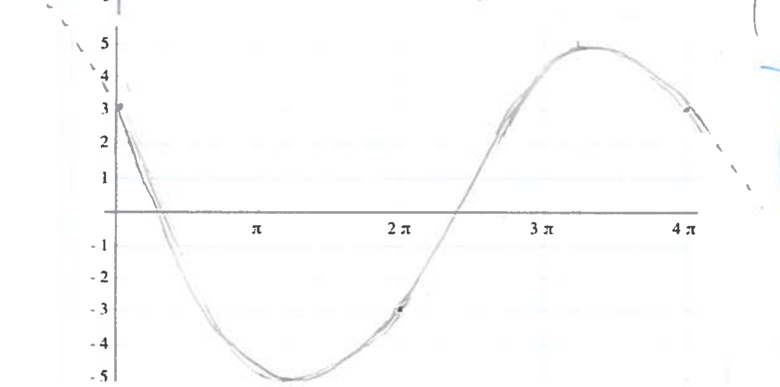


find ω: $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{8}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

if $\omega = 1/2$, then one period takes $2 \times 2\pi$ s. or 4π s.

Amplitude using c)

$$\left(3^2 + \frac{2^2}{(1/2)^2}\right)^{1/2} = (9 + 16)^{1/2} = 5 = A$$



if v_0 was negative, then this is the plot.

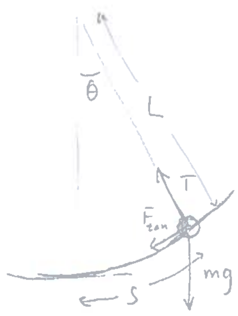
2. What should the damping value be in terms of L, g, and m in order to critically damp a pendulum? (you might be able to just write the answer down if you recalled the natural frequency of a simple pendulum, but what fun is that? please work through the following:

a. Set up a force equation for a simple, damped pendulum of length L, mass m, in regular earth gravity. (assuming small angles of oscillation)

b. Use the function $x = e^{\alpha t}$ as a test solution and find the relationship between α and the parameters of the pendulum.

c. Identify the case where the damping is critical and use that to express the value of your damping term in terms of L, g, and m.

a)



if θ is small, $\sin\theta \approx \theta$; then $s = L\theta$

$$F_{tangential} = -mg\sin\theta = -\frac{mg}{L}s$$

2nd law

$$\Sigma F = \underbrace{-b\dot{s}}_{\text{damping}} \underbrace{- \frac{mg}{L}s}_{\text{restoring force}} = ma = m\ddot{s}$$

Damped, simple pendulum

b) try $s = e^{\alpha t}$ in $\ddot{s} + \frac{b}{m}\dot{s} + \frac{g}{L}s = 0$

$$\frac{ds}{dt} = \alpha e^{\alpha t}$$

$$\frac{d^2 s}{dt^2} = \alpha^2 e^{\alpha t}$$

$$\alpha^2 e^{\alpha t} + \frac{b}{m}\alpha e^{\alpha t} + \frac{g}{L}e^{\alpha t} = 0$$

$\vdots$

$$\alpha^2 + \frac{b}{m}\alpha + \frac{g}{L} = 0$$

$\Rightarrow$ solve for α using Quadratic $\underbrace{\frac{-b \pm \sqrt{b^2 - 4ac}}{2}}$

too many b's ☺

$b = \frac{b}{m}$

$a = 1$

$c = \frac{g}{L} = \omega^2$

$$\alpha = \frac{-b/m \pm \sqrt{b^2/m^2 - 4\frac{g}{L}}}{2}$$

Oh, but if $\beta = \frac{b}{2m}$, then $= \frac{-2\beta \pm \sqrt{4\beta^2 - 4\frac{g}{L}}}{2}$

$$\alpha = -\beta + \sqrt{\beta^2 - \frac{g}{L}}$$

c) damping will be critical when $\beta^2 - \frac{g}{L} = 0$, then $\beta = \sqrt{\frac{g}{L}}$

or $\frac{b}{2m} = \sqrt{\frac{g}{L}}$

$$\therefore \quad b = 2m\sqrt{\frac{g}{L}}$$

* note, m does appear here, since a larger mass will require more damping to stop it.