Ellipse Math (Gravity)

$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{U_{eff}}$$

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r}$$

[ℓ = angular momentum (constant)]

$$E = \underbrace{\frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r}}_{\text{at perihelion, } r_p \text{ \& aphelion, } r_a}$$

$$\downarrow$$

$$\dot{r} = 0$$

$$E = \frac{\ell^2}{2\mu r_a^2} - \frac{Gm_1m_2}{r_a} \quad \text{or} \quad E = \frac{\ell^2}{2\mu r_a^2} - \frac{GM\mu}{r_a}$$

[$\mu = \frac{m_1m_2}{m_1+m_2} = \frac{m_1m_2}{M}$]

and

$$E = \frac{\ell^2}{2\mu r_p^2} - \frac{GM\mu}{r_p}$$

solve these two simultaneously (eliminate ℓ)

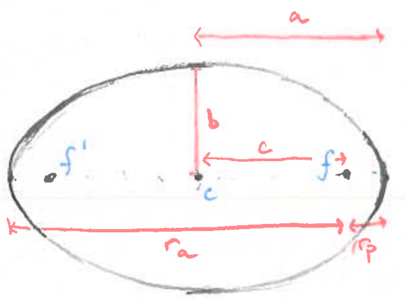
$$Er_a^2 = \frac{\ell^2}{2\mu} - GM\mu r_a$$

$$-\left(Er_p^2 = \frac{\ell^2}{2\mu} - GM\mu r_p\right)$$

$$E(r_a^2 - r_p^2) = 0 - GM\mu(r_a - r_p)$$

$$E = -GM\mu\frac{(r_a - r_p)}{(r_a^2 - r_p^2)} = -GM\mu\frac{r_a - r_p}{(r_a - r_p)(r_a + r_p)} = \frac{-GM\mu}{r_a + r_p} = \frac{-GM\mu}{2a}$$

] Energy in terms of G and orbit masses and semi-major axis.



$r_a + r_p = 2a$

$a\epsilon = c$

$\epsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2}$

$b = \sqrt{r_a r_p}$

$$\boxed{E = -\frac{GM\mu}{2a}}$$ Energy

$$\text{Eccentricity } \epsilon = \sqrt{1 - \left(\frac{b}{a}\right)^2} \longrightarrow \epsilon = \sqrt{1 - \frac{-\ell^2}{2\mu E} \times \frac{4E^2}{G^2M^2\mu^2}} = \boxed{\sqrt{1 + \frac{2E\ell^2}{G^2M^2\mu^3}} = \epsilon}$$

Eccentricity

solve the 2 E equation again, but this time keep E & ℓ

$$\left.\begin{aligned} Er_a &= \frac{\ell^2}{2\mu r_a} - GM\mu \\ -\left(Er_p \right. &= \left.\frac{\ell^2}{2\mu r_p} - GM\mu\right) \end{aligned}\right\} = E(r_a - r_p) = \frac{\ell^2}{2\mu r_a} - \frac{\ell^2}{2\mu r_p} = \frac{\ell^2}{2\mu}\left(\frac{1}{r_a} - \frac{1}{r_p}\right) = \frac{-\ell^2}{2\mu}\left(\frac{r_a - r_p}{r_a r_p}\right) \Rightarrow r_a r_p = \frac{-\ell^2}{2\mu E} = b^2$$

Algebra for $t(r)$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{GM\mu}{r}$$] solve for $\dot{r} = \frac{dr}{dt}$

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E + \frac{GM\mu}{r} - \frac{\ell^2}{2\mu r^2}\right)} = \pm\sqrt{\frac{2}{\mu r^2}\left(Er^2 + GM\mu r - \frac{\ell^2}{2\mu}\right)}$$

$$\therefore \int dt = \pm\sqrt{\frac{\mu}{2}} \int_{r_0}^{r} \frac{r\,dr}{\left(Er^2 + GM\mu r - \frac{\ell^2}{2\mu}\right)^{1/2}}$$ ... let $X = cx^2 + bx + a$ $\left\{ \begin{array}{l} c = E \\ b = GM\mu \\ a = -\frac{\ell^2}{2\mu} \end{array} \right\}$

$$\int dt = \pm\sqrt{\frac{\mu}{2}} \underbrace{\int \frac{x\,dx}{\sqrt{X}}}_{\text{from integral tables}}$$

we will evaluate between r_a & r_p.

at r_a & $r_p \Rightarrow X = 0$

because $\dot{r} = 0$

$$= \underbrace{\frac{\sqrt{X}}{c}}_{=0} - \frac{b}{2c} \underbrace{\int \frac{1}{\sqrt{X}}\,dx}_{\frac{-1}{\sqrt{-c}} \sin^{-1}\left(\frac{2cx+b}{\sqrt{-(4ac-b^2)}}\right)}$$

$$= \pm\sqrt{\frac{\mu}{2}}\left(-\frac{GM\mu}{2(E)}\left(\frac{-1}{\sqrt{-E}}\left(\sin^{-1}\left(\frac{2Er + GM\mu}{\sqrt{4\frac{\ell^2}{2\mu}E + G^2M^2\mu^2}}\right)\right)\right)\right]_{r_p}^{r_a}$$

again, since $X = 0$ @ $r = r_a, r_p$, we can use $0 = Er_a^2 + GM\mu r_a - \frac{\ell^2}{2\mu}$

and employ the Quadratic formula to solve for r_a (or r_p)

$$r_{a,p} = \frac{-GM\mu \pm \sqrt{G^2M^2\mu^2 + 4E\frac{\ell^2}{2\mu}}}{2E}$$

$$\underbrace{2Er_{a,p} + GM\mu}_{\text{numerator}} = \pm\underbrace{\sqrt{G^2M^2\mu^2 + 4E\frac{\ell^2}{2\mu}}}_{\text{denominator}}$$ } compare this to the terms inside the $\sin^{-1}(\,)$ after the integration.

so,

$$t(\text{peri}\to\text{ap}) = \pm\sqrt{\frac{\mu}{2}}\left(\frac{GM\mu}{2E\sqrt{-E}}\right)\left(\underbrace{\sin^{-1}(+1)}_{=\frac{\pi}{2}} - \underbrace{\sin^{-1}(-1)}_{=-\frac{\pi}{2}}\right) = GM\left(\frac{\mu}{-2E}\right)^{3/2}\pi$$

but, from Earlier, $E = -\frac{GM\mu}{2a}$, so $t_{(p\to a)} = GM\pi\left(\frac{\mu}{-2\left(-\frac{GM\mu}{2a}\right)}\right)^{3/2} = \frac{\pi}{\sqrt{GM}}\, a^{3/2}$ } this is ½ the period so, times 2 to get total.

$$\boxed{t = \frac{2\pi}{\sqrt{GM}}\, a^{3/2}}$$ $\Rightarrow$ Kepler's 3rd Law.

now, find $r(\varphi)$

$$\dot r = \pm\sqrt{\frac{2}{\mu}}\sqrt{\left(E + \frac{GM\mu}{r} - \frac{\ell^2}{2\mu r^2}\right)} \qquad \dot\varphi = \frac{\lambda}{\mu r^2} \qquad \frac{dr}{d\varphi} = \frac{dr}{dt}\frac{dt}{d\varphi} = \frac{\dot r}{\dot\varphi}$$

$$\frac{dr}{d\varphi} = \frac{\pm\sqrt{\frac{2}{\mu}}\sqrt{E+\frac{GM\mu}{r} - \ell^2/2\mu r^2}}{\ell/\mu r^2} = \pm\sqrt{\frac{2\mu}{\ell^2}}\, r^2\sqrt{E+\frac{GM\mu}{r}-\frac{\ell^2}{2\mu r^2}}$$

$$\int d\varphi = \pm\frac{\ell}{\sqrt{2\mu}}\int \frac{1}{r^2}\,\frac{1}{\frac{1}{r}\sqrt{Er^2+GM\mu r - \frac{\ell^2}{2\mu}}}\,dr = \pm\frac{\ell}{\sqrt{2\mu}}\int\frac{dr}{r\left(Er^2+GM\mu r - \frac{\ell^2}{2\mu}\right)^{1/2}}$$

$\downarrow$

let $X = cx^2+bx+a$ $\quad \begin{cases} c=E\\ b=GM\mu \\ a = -\ell^2/2\mu\end{cases}$

$$\int\frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bx+2a}{x\sqrt{-(4ac-b^2)}}\right)$$

$$\int_{\varphi_0}^{\varphi} d\varphi = \underbrace{\pm\frac{\ell}{\sqrt{2\mu}}\cdot\frac{1}{\sqrt{\ell^2/2\mu}}}_{=1}\,\sin^{-1}\left(\frac{GM\mu r + 2\left(-\frac{\ell^2}{2\mu}\right)}{r\sqrt{G^2M^2\mu^2+4\frac{\ell^2}{2\mu}E}}\right)\Bigg]^{r}$$

use $\epsilon = \sqrt{1+\frac{2E\ell^2}{G^2M^2\mu^3}}$

or $GM\mu\epsilon = \sqrt{G^2M^2\mu^2+\frac{2E\ell^2}{\mu}}$ to rewrite as

$$\varphi-\varphi_0 = 1\times\sin^{-1}\left(\frac{GM\mu^2 r - \ell^2}{\epsilon GM\mu^2 r}\right)\Bigg]$$

$$\sin(\varphi-\varphi_0) = \frac{GM\mu^2 r - \ell^2}{\epsilon GM\mu^2 r}$$ $\Big\}$ solve for r

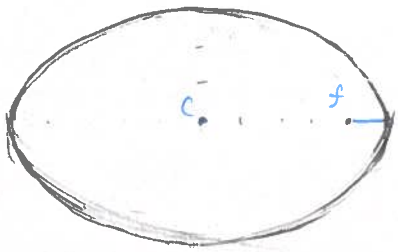
$$-\epsilon GM\mu^2 r\sin(\varphi-\varphi_0) + GM\mu^2 r = \ell^2$$

$$rGM\mu^2\left(1-\epsilon\sin(\varphi-\varphi_0)\right) = \ell^2$$

$$r = \frac{\ell^2/GM\mu^2}{1-\epsilon\sin(\varphi-\varphi_0)}$$

$$\boxed{r = \frac{1}{GM\mu^2}\,\frac{\ell^2}{(1+\epsilon\cos\varphi)}}$$

Ellipse as $r(\varphi)$



make perihelion $\varphi_0 = \pi/2$

thus $\sin(\varphi-\pi/2) = -\cos(\varphi)$