Special Relativity

1. Early Hints

Galilean Invariance & Reference Frames

[Sim from https://sciencesims.com/sims/reference-frame-linear-V]

Faraday's Giant Electromagnet

1830's: Michael Faraday was making charges move with coils and magnets.

He experimentally showed that a changing magnetic field will create a current, and the inverse. These phenomena we now call induction.

1855: Maxwell: "On Faraday's lines of force"

"We can scarcely avoid the conclusion that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena."

Maxwell, ~1862

J. C. Maxwell

The Maxwell Equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}\right)$$

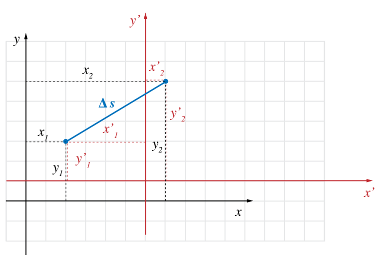
These were the laws the described the electromagnet phenomena, just like $F = ma$ described mechanical motions.

1.1 Invariance

Answer to the question: what do observers in two difference inertial reference frames agree on?

For Newton's Laws: Length!

$$\Delta r^2_{\text{Eucl.}} = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \geqslant 0 \tag{1}$$



Distance in Galilean Relativity

Simple Example (in 2d)

Unprimed Frame

$$\sqrt{(6-3)^2 + (7-2)^2}$$

Primed Frame

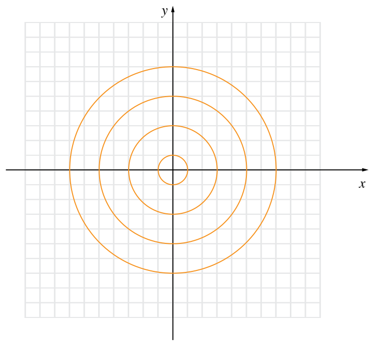
$$\sqrt{(1-(-4)^2 + (5-2)^2}$$

It was quickly realized that the Maxwell equations did not share the same invariance as Newton's Laws.

This was the first problem.

And if there is a translational invariance, then one *could* determine whether or not a frame had a velocity.

[Sim from https://sciencesims.com/sims/soundwaves-cyl]



Position of wave traveling at speed c after time t:

$$r=\sqrt{x^2+y^2}=ct$$

So, we can say:

$$x^2+y^2=c^2t^2$$

or:

$$-c^2t^2+x^2+y^2=0$$

We could try to change to a new coordinate frame using the Galilean Transformation:

$$x'=x-vt$$

and

$$t'=t$$

(we'll just do it in 1d for simplicity) So, does:

$$-c^2t'^2+x'^2\stackrel{?}{=}0$$

We can check by replacing x' with its Galilean Transform:

$$x'=x-vt$$

You can do some algebra and show that

$$-c^2t'^2+x'^2=vt(vt-2x)\neq 0$$

That's all fine for a water wave, experimentally speaking, you can measure easily see if you're moving relative to a water wave. But, as experiments showed...

The second 'problem' grew out of several experiments that showed that light wasn't acting like a 'regular' wave would.

Normally, a *mechanical wave* would be affected by the motion of the medium it's traveling in. i.e. a water wave in a moving current will take on the velocity of the current as well as its own wave speed.

Such affects were found to be non-existent through several experiments.

Experiments:

- 1851: Fizeau experiment (speed of light in a moving fluid)
- 1887: Michelson–Morley experiment (speed of light depending on which way Earth is moving.)

Lorentz

Notably, in response to experiments, Lorentz and others proposed the following *transforms*, to be used instead of the Galilean transformations.

$$\begin{aligned}
x &= \gamma\left(x'+\beta ct'\right) && (2)\\
y &= y' && (3)\\
z &= z' && (4)\\
ct &= \gamma\left(ct'+\beta x'\right) && (5)
\end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\beta = \frac{v}{c}$

1.2 Wording: What's a postulate?

postulare: to ask, demand; claim; require,

This is different than an *assumption*. It's more serious. The most famous postulate might be Euclid's #5: parallel lines don't cross.

2. Einstein's Postulates:

1. The laws of electrodynamics and optics will be valid for all frames of reference for which equations of mechanics hold good.
2. Light in a vacuum moves at a speed c which is independent of the state of motion of the emitting body.

1905
On the Electrodynamics of Moving Bodies

ANNALEN
DER
PHYSIK.

F. A. C. GREN, L. W. GILBERT, J. C. POGGENDORFF, G. UND E. WIEDEMANN.
VIERTE FOLGE.
BAND 17.
KURATORIUM:
F. KOHLRAUSCH, M. PLANCK, G. QUIN
W. C. RÖNTGEN, E. WARBURG.
UNTER MITWIRKUNG
DER DEUTSCHEN PHYSIKALISCHEN GESELLS
M. PLANCK
PAUL DRUDE.
MIT FÜNF FIGURENTAFELN.
LEIPZIG, 1905.
VERLAG VON JOHANN AMBROSIUS BAR

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3. *Zur Elektrodynamik bewegter Körper;*
von A. Einstein.

Daß die Elektrodynamik Maxwells — wie dieselbe gegenwärtig aufgefaßt zu werden pflegt — in ihrer Anwendung auf bewegte Körper zu Asymmetrien führt, welche den Phänomenen nicht anzuhaften scheinen, ist bekannt. Man denke z. B. an die elektrodynamische Wechselwirkung zwischen einem Magneten und einem Leiter. Das beobachtbare Phänomen hängt hier nur ab von der Relativbewegung von Leiter und Magnet, während nach der üblichen Auffassung die beiden Fälle, daß der eine oder der andere dieser Körper der bewegte sei, streng voneinander zu trennen sind. Bewegt sich nämlich der Magnet und ruht der Leiter, so entsteht in der Umgebung des Magneten ein elektrisches Feld von gewissem Energiewerte, welches an den Orten, wo sich Teile des Leiters befinden, einen Strom erzeugt. Ruht aber der Magnet und bewegt sich der Leiter,

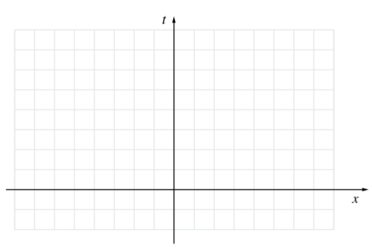
Einstein's Paper

Einstein's paper took all these pieces and assembled them into a new understanding of our physical world:

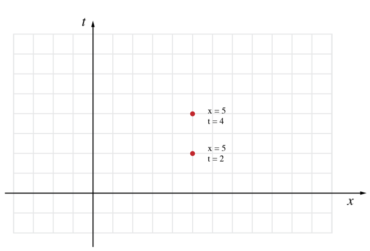
[On the Electrodynamics of Moving Bodies (pdf)]

But, 3 years later, an extension on the idea was discussed by Herman Minkowski.

Original Minkowski: [Space and Time]



A Blank Space time diagram



A Galilean transformation example

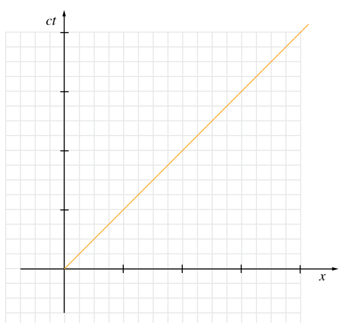
[Sim from https://sciencesims.com/sims/coordinate-transforms/]

2.3 Galilean Transform in Linear Algebra

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \tag{6}$$

[Sim from https://sciencesims.com/sims/shear-transform/]

3. Space Time Diagrams



Construct our moving frame axes

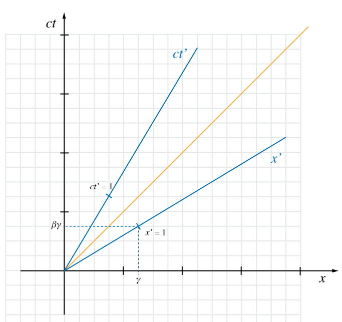
Let's make our primed axes: x' and ct'

Start by finding the point along x' where $x' = 1$ and $ct' = 0$

Using the transforms above (2) and (5), we can say that our primed axis x' will pass through the point: $x = \gamma$ and $ct = \beta\gamma$. This means the slope of the x' axis in the x frame is equal to β.

For simplified drawing we can say our $v = 0.6c$, thus $\gamma = 1.25$

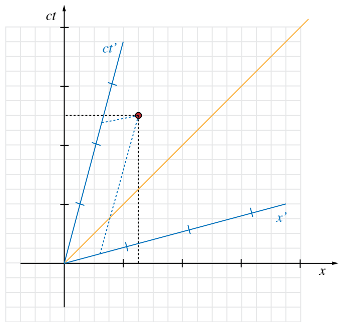
Doing the same for the ct' axis, we arrive at a slope of $\frac{1}{\beta}$ for that line.



The diagram with the x' and ct' axes in place

Now we have our primed axes in place. You can see that they are symmetric about the $ct = x$ line. That line represents the world line of a photon emitted from the origin. It has the speed of c.

Now we can add an **event** to our diagram. It will have two sets of coordinates. One for *stationary* reference frame, and one for the moving frame. (Of course, the whole point here is that one doesn't deserve to be called stationary any more than the other, but we'll say we are located in the unprimed coordinate system, and thus stationary w.r.t to it.)



An event shown in both frames

[Sim from https://sciencesims.com/sims/minkowski/]

3.4 Minkowski Invariant

In this new spacetime (as opposed to Euclidian Space), we still can have an invariant quantity:

$$-c^2\Delta t'^2 + \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = -c^2\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{7}$$

or more simply in 1+1 dimensions:

$$-c^2\Delta t'^2 + \Delta x'^2 = -c^2\Delta t^2 + \Delta x^2$$

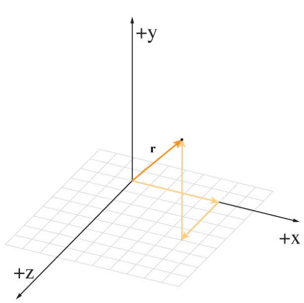
These can be shown to be invariant under a Lorentz Transformation

Using:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{8}$$
$$x' = \gamma\left(x - vt\right) \tag{9}$$
$$y' = y \tag{10}$$
$$z' = z \tag{11}$$

Expand the squares and group like terms and you will see that indeed the Minkowski Line Element is preserved during a Lorentz Transformation.

4. What is ds^2 ?



$\mathbb{R}^3$

4.5 $\mathbb{R}^3$

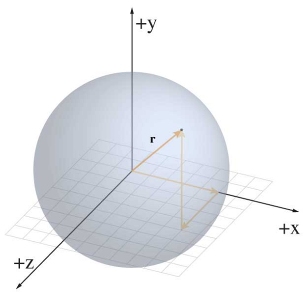
Our everday 3d world is known more formally as

$$\mathbb{R}^3$$

. This is a vector space with 3 real coordinates. We map them onto our *human* notions of x, y, z.

We describe positions using regular vectors:

$$\mathbf{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$



The collection of places where r^2 equals C

We can express the length of such a vector by the following:

$$|\mathbf{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2} \tag{12}$$

or, if that was some change in position, then we could just as easily consider:

$$\Delta\mathbf{r}^2 = \Delta r_x^2 + \Delta r_y^2 + \Delta r_z^2 \tag{13}$$

And using galilan transforms between coordinate systems, everyone would 'agree' that that value of $\Delta\mathbf{r}^2$ remains the same.

Furthermore, we can image the region of space where that quantity is equal to some value: it's a sphere.

[Sim from https://sciencesims.com/sims/transforms-pathlength/]

4.6 in Minkowski 4-space

Recall the 2nd postulate - speed of light is the same

Thus, for a photon:

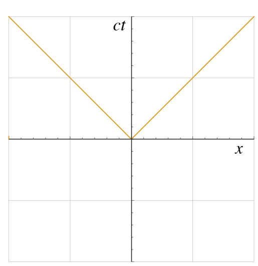
$$c^2 dt^2 = dx^2 + dy^2 + dz^2$$

Or, if we're just concerned with one dimensional motion:

$$c^2 dt^2 = dx^2 \tag{14}$$

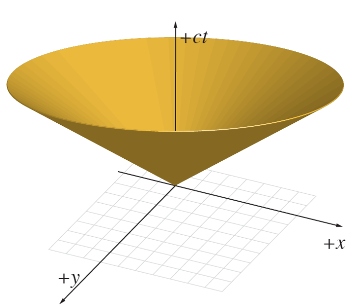
$$-c^2 dt^2 + dx^2 = 0 \tag{15}$$

(And every observer, if they're going with the 2nd postulate would have the same result)



World lines for massless particles moving at speed c.

In 1d, we can plot ct(x) and see that we get two lines leading away from the origin at 45° These are the world lines of photons.



2d light cones

In 2 spatial dimensions, we can see that these lines are really just the edges of a cone – the light cone.

Hence,

$$ds^2 = -c^2 dt^2 + dx^2 \tag{16}$$

will be our invariant quantity for spacetime.

Now, what if that 'separation' in spacetime $\neq 0$?

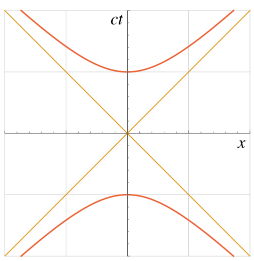
i.e. what if

$$ds^2 = -c^2 dt^2 + dx^2 > 0$$

or

$$ds^2 = -c^2 dt^2 + dx^2 < 0$$

The startling difference between this and the distance measurement in $\mathbb{R}_3$ is that now we can have positive *and negative* values.



The hyperbolas

Let's start with $ds^2 < 0$, eg:

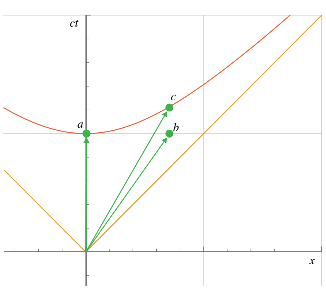
$$ds^2 = -c^2 dt^2 + dx^2 = -1$$

We can plot these separations using:

$$ct = \sqrt{dx^2 + 1}$$

And all the points along this hyperbola are the same 'spacetime' distance away from us.

These points would all be considered 'time-like' in the separations from the origin.

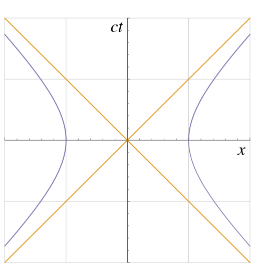


3 events, *a*,*b*, and *c*

Consider the 3 events shown in the figure.

a. A ball in your hand t seconds from now.
b. A ball over there t seconds from now.
c. A ball over there $> t$ seconds from now.

Which two have the same spacetime separation?



The space-like hyperbola

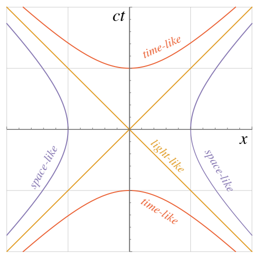
We can also have $ds^2 > 0$, eg:

$$ds^2 = -c^2 dt^2 + dx^2 = +1$$

We can plot these separations using:

$$ct = \pm\sqrt{dx^2 - 1}$$

And all the points along this hyperbola are the same 'spacetime' distance away from us.

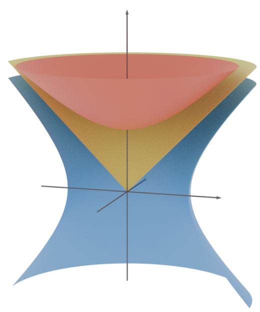


The space-like hyperbola

Space-like separations: these are events we can have no causal influence over.

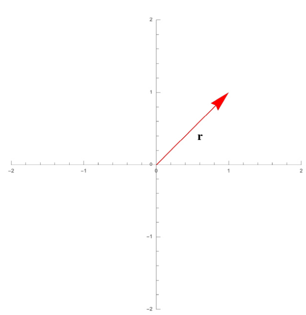
Time-like separations: if we can travel fast enough, or send signals of light, then we can causally influence these events.

Light-like: the worlds lines of photons.



Spacetime in 2+1 dimensions.

Other Transforms: Rotations



A vector $\mathbf{r}$

Shown is a vector $\mathbf{r}$

$$\mathbf{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

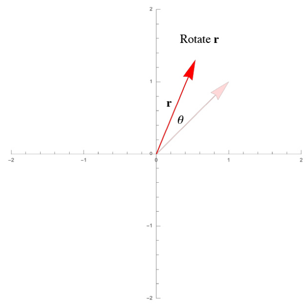
If we wanted to rotate it, we could *transform* the vector components:

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \\ z &= z' \end{aligned}$$

We can express a rotation about an axis as a multiplication of the vector $\mathbf{r}$ with a Rotation Matrix $\mathcal{R}$

$$\mathcal{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{r}'$$



A vector $\mathbf{r}$ that has been rotated.

Our Lorentz transformations can also be expressed in a matrix form

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Let

$$\gamma \equiv \cosh\xi \geqslant 1$$

and then since $\cosh^2\xi - \sinh^2\xi = 1$:

$$\gamma\beta = \sinh\xi$$

The Lorentz transformations are analogous to a hyperbolic rotation between coordinate frames.

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

[Sim from https://sciencesims.com/sims/world-lines/]

Here we see several spactime travelers depart from the origin at different speeds, appreciable fractions of c. If they're clocks ticked every hour (or whatever unit is shown), then the placement of those ticks follows the hyperbolas seen in the previous graphs.

[Sim from https://sciencesims.com/sims/world-lines-sequencer/]

Here we see several spactime travelers depart from the origin at different speeds, appreciable fractions of c. If they're clocks ticked every hour (or whatever unit is shown), then the placement of those ticks follows the hyperbolas seen in the previous graphs.