Special Orbits

1. Why Orbits?

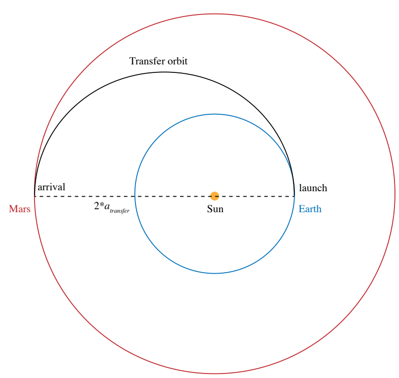
1.1 Bertrand's Theorem

The only central-force potentials $U(r)$ for which all bounded orbits are closed are:

1. Gravity: $U \propto \frac{1}{r}$
2. Springs: $U \propto r^2$

2. Special Orbits

2.1 Hohmann Transfer Orbit



A typical Hohmann Transfer Orbit

The semi-major axis of the transfer orbit:

$$a_{\text{transfer}} = \frac{r_e + r_m}{2} \tag{1}$$

2.2 2 -> 3 bodies

3. Lagrange Points

3.3 Preliminaries

Any conservative force can be related to potential energy by the following (in 1-D):

$$F(x) = -\frac{dU}{dx} \tag{2}$$

or in 3-D:

$$\mathbf{F} = -\nabla U \tag{3}$$

For example, mass on a spring:

$$F_x = -kx \tag{4}$$

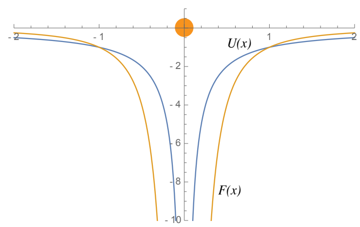
$$U_{\text{spring}} = \frac{1}{2}kx^2 \tag{5}$$

Or, for gravity near the surface of the Earth:

$$F_y = -mg \tag{6}$$

$$U_{\text{grav}} = mgy \tag{7}$$

3.4 Gravitational Potential



Potential Energy and Force due to a single gravitational source.

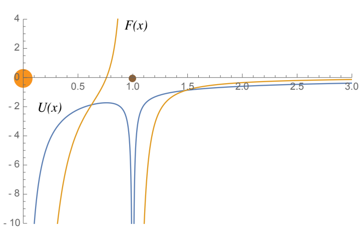
And, if we consider any distance away from the Earth (or central body), we would have:

$$F(r) = -G\frac{M_E m}{r^2} \tag{8}$$

$$U(r) = -G\frac{M_E m}{r} \tag{9}$$

Binary Systems

Potential Landscape

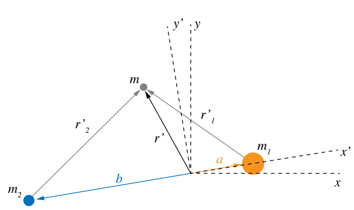


Adding another source makes a more complicated landscape

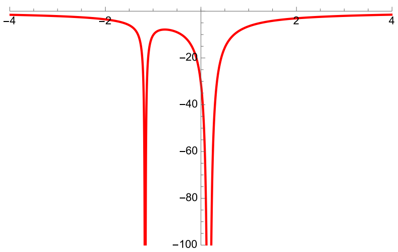
Now, if we have 2 source bodies, the functions have more features. Now, we ask what happens to a 3rd body in this landscape.

Restricted 3 body

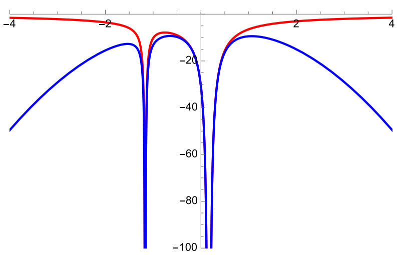
- m_1 much greater than m_2
- Circular Orbits
- m_3 is negligible in mass



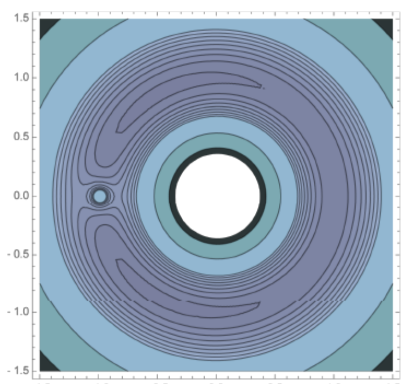
Coordinates for a restricted 3 body system



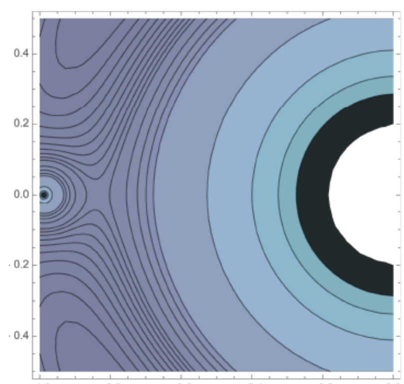
The potential (non-rotating)



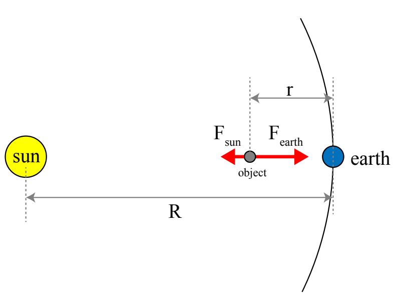
The potential (rotating)



contour plot



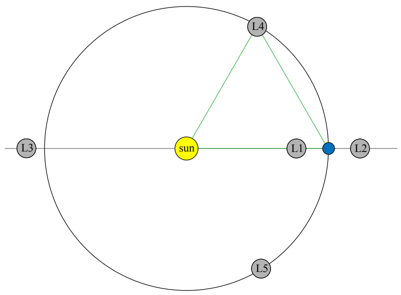
contour plot



The basic free body diagram showing the forces on a small body near two larger ones.

The basic FBD and L1 point

There are five Lagrange points. Some are interesting for technology applications.



All 5 Lagrange points

3.5 Find L1

The sum of forces acting on the earth is equal to its mass time the acceleration, which for an object moving in a circular orbit is centripetal and equal to v^2/R:

$$\Sigma F_{\text{on Earth}} = \frac{GMm_E}{R^2} = \frac{m_E v^2}{R}$$

Thus we can say:

$$\frac{GM}{R} = v^2$$

but, based on the relation between speed and period, T:

$$v = \frac{2\pi R}{T}$$

or

$$v^2 = \frac{4\pi^2 R^2}{T^2}$$

So we can rewrite as:

$$\frac{GM}{R} = \frac{4\pi^2 R^2}{T^2}$$

Rearranging yields Kepler's Third Law

$$\frac{GM}{R^3} = \frac{4\pi^2}{T^2} \tag{10}$$

Now consider the sum of forces on the satellite:

$$\Sigma F_{\text{on Sat}} = \frac{GMm_{\text{sat}}}{(R-r)^2} - \frac{Gm_E m_{\text{sat}}}{r^2} = \frac{m_{\text{sat}} v_{\text{sat}}^2}{(R-r)}$$

This we can simplify to, using Kepler's Third law for the satellite:

$$\frac{GM}{R-r} - \frac{Gm_E(r-R)}{r^2} = v_{\text{sat}}^2 = \frac{4\pi^2(R-r)^2}{T_{\text{sat}}^2}$$

or

$$\frac{GM}{(R-r)^3} - \frac{Gm_E}{r^2(R-r)} = \frac{4\pi^2}{T_{\text{sat}}^2} \tag{11}$$

Lastly, since we want their periods to be the same: $T_{\textbf{sat}} = T$

$$\frac{GM}{(R-r)^3} - \frac{Gm_E}{r^2(R-r)} = \frac{GM}{R^3} \tag{12}$$

If we put in the masses of the two major bodies, the Earth and Sun for example, we can calculate the value of r in terms of R. For our earth-sun system, the L1 point would be located at

$$r = 0.009969 \text{ AU}$$

or about 1/100 of the earth-sun distance. Changing signs in the sums of forces could allow to calculate the L2 and L3 positions as well.