# Lagrangian - Part 2



## 1. Generalized Momenta

For a simple, free particle, the kinetic Energy is:

$$T=\frac{1}{2}m\dot{x}^2 \tag{1}$$

Take the derivative of $L$ w.r.t $\dot{x}$ :

$$\frac{\partial L}{\partial \dot{x}}=m\dot{x} \tag{2}$$

This looks like *momentum*.

Thus, for generalized coordinates $q_k$, we can also have a **generalized momentum**

$$p_k=\frac{\partial L}{\partial \dot{q}_k} \tag{3}$$

The Lagrangian equations can then be written as simply;

$$\frac{dp_k}{dt}=\frac{\partial L}{\partial q_k} \tag{4}$$

But what if a particular Lagrangian is missing one of the $q_k$ dependencies?

### 1.1 Conservation of ...

In that case, we can quickly see that the generalized momenta of that coordinate doesn't change w.r.t time:

$$\frac{dp_k}{dt}=\frac{\partial L}{\partial q_k}=0 \tag{5}$$

Which implies that that particular generalized momentum is a **conserved** quantity!

(Such a coordinate is called **cyclic** or **ignorable**)

### 1.2 2d with central force

The Lagrangian for a particle confined to a plane with a central force was:

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-U(r,\theta) \tag{6}$$

The 2 coordinates required 2 Lagrange Equations

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \tag{7}$$

and

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \tag{8}$$

Evaluating these results in two acceleration terms: $a_r$, and $a_\theta$

$$a_r = \ddot{r} - r\dot{\theta}^2 \tag{9}$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{10}$$

## 1.3 Central Force, with a Spring

Let's put a specific force in there. How about a spring? The potential of the spring is easy to write:

$$U_{\text{sp}} = \frac{1}{2}kr^2 \tag{11}$$

Thus, the Lagrangian becomes:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}kr^2 \tag{12}$$

This will affect the outcome of the $r$ equation:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -kr \tag{13}$$

The momentum in the $\theta$ coordinate

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \tag{14}$$

(Which is really just the angular momentum)

But, because $\theta$ is a cyclic coordinate, then we know that angular momentum is conserved.

$$\frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta} = 0 \tag{15}$$

therefore:

$$p_\theta = \textbf{constant} \tag{16}$$

Solving these two equation simulataneously (i.e. eliminating $\dot{\theta}$)

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -kr \tag{17}$$
$$p_\theta = mr^2\dot{\theta} \tag{18}$$

leads to:

$$\ddot{r} - \frac{p_\theta^2}{m^2r^3} + \omega_0^2 r = 0 \tag{19}$$

where $\omega_0 = \sqrt{k/m}$ as usual.

What does this do for us?

$$m\ddot{r} = \underbrace{\frac{p_\theta^2}{mr^3} - m\omega_0^2 r}_{\text{position only!}} \tag{20}$$

Normally, contributions that are position dependent only, can considered as some sort of potential.

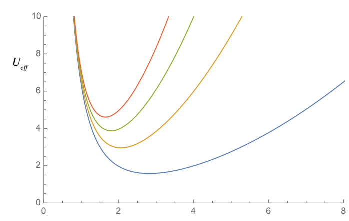
Thus, we can introduce an **effective** potential: $U_{\text{eff}}$

$$\frac{p_\theta^2}{mr^3} - m\omega_0^2 r \equiv -\frac{dU_{\text{eff}}}{dr} = F(r) \tag{21}$$

$$U_{\text{eff}}(r) = -\int^r F(r)dr = -\int^r \left(\frac{p_\theta^2}{mr^3} - m\omega_0^2 r\right) dr \tag{22}$$

$$= \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2 \tag{23}$$

The second term is recognizable as the potential from the spring, but the first term is really a bit of the kinetic energy that only depends on position. We lump those two together and call it an effective potential.




$U_{\text{eff}}$ plots for different $k$ values.

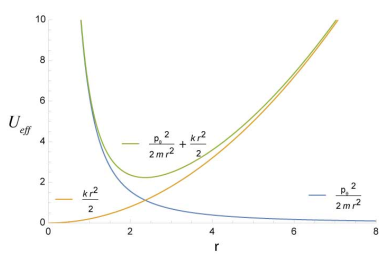
Plots of

$$U_{\text{eff}}(r) = \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2$$

for different $k$ values.

What does it mean when $\frac{dU}{dr} = 0$?

These are equilibrium points in $r$. (i.e. $\ddot{r} = 0$). Small displacements will lead to oscillations about the equilibrium points.




The summation of two potentials

## 1.4 Oscillations around equilibrium

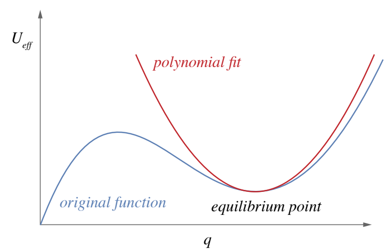
If we expand our $U_{\text{eff}}$ using Taylor:

$$U_{\text{eff}}(q) = U_{\text{eff}}(q_0) + \left.\frac{dU_{\text{eff}}}{dq}\right|_{q_0} + \frac{1}{2!}\left.\frac{d^2U_{\text{eff}}}{dq^2}\right|_{q_0}(q-q_0)^2 + \cdots \tag{24}$$

The third term looks like a *Harmonic Oscillator Potential:*

$$U_{\text{H.O.}} = \frac{1}{2}k_{\text{eff}}(q-q_0)^2 \tag{25}$$

where we have a new *effective spring constant:* $k_{\text{eff}} = U''_{\text{eff}}(q_0)$




An effective potential with an stable equilibrium point.

The frequency of small oscillations there is then just:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{U''_{\text{eff}}(q_0)}{m}} \tag{26}$$

Thus for our mass/spring 2d system:

$$U_{\text{eff}} = \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2 \tag{27}$$

The first derivative: $U'_{\text{eff}}$ is

$$U'_{\text{eff}} = -\frac{p_\theta^2}{mr^3} + kr \tag{28}$$

and thus our equilibrium value for $r$ is where this zero, or:

$$r = r_0 = \left(\frac{p_\theta^2}{mk}\right)^{(1/4)} \tag{29}$$

The second derivative:

$$U''_{\text{eff}}(r) = \frac{3p_\theta^2}{mr^4} + k \tag{30}$$

Evaluate this at $r = r_0$ :

$$U''_{\text{eff}}(r_0) = \frac{3p_\theta^2}{m\frac{p_\theta^2}{mk}} + k \tag{31}$$

Thus

$$k_{\text{eff}} = 4k \tag{32}$$

Thus, our frequency about the equilibrium position is:

$$\omega = \sqrt{\frac{U''_{\text{eff}}(r_0)}{m}} = \sqrt{\frac{4k}{m}} = 2\omega_0 \tag{33}$$

Now, compare to rotational frequency:

$$\omega_{\text{rot}} = \frac{v_\theta}{r_0} = \frac{p_\theta/mr_0}{r_0} = \frac{p_\theta}{mr_0^2} \tag{34}$$

Use the equilibrium radius:

$$r_0 = \left(\frac{p_\theta^2}{mk}\right)^{(1/4)} \tag{35}$$

The rotation frequency is:

$$\omega_{\text{rot}} = \frac{p_\theta}{mr_0^2} = \frac{p_\theta}{\sqrt{\frac{mp_\theta^2}{k}}} = \sqrt{\frac{k}{m}} = \omega_0 \tag{36}$$

So, the rotational frequency is the same as the natural frequency of the spring/mass:

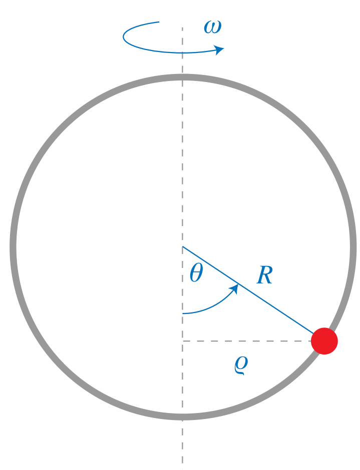
$$\omega_{\text{rot}} = \omega_0 \tag{37}$$

Therefore the radial oscillations are twice that of the rotational frequency, which implies that the orbits are *closed*, meaning it will return to the starting point after each rotation.

2d spring Simulation

## 2. More Examples

Example Problem #1:




Bead constrained to a loop

A bead with mass $m$ is constrained to a circular hoop that rotates around the vertical axis with speed $\dot{\phi} = \omega$. The bead's position is described by the angle $\theta$. Describe this system by solving the Lagrangian and noting any interesting aspects, i.e. equilibriums.

First, to construct the Lagrangian, we need the kinetic and potential energy of the bead. The bead is free to move about the radius of the hoop, which will have a speed of $R\dot{\theta}$. The other component of the bead's velocity will be due to the rotation about the vertical axis. Based on the figure, this is just $\rho\omega = R\sin\theta\omega$. Putting these together to calculate the square of the velocity:

$$v^2 = R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta \tag{38}$$

Thus the kinetic energy will be:

$$T = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) \tag{39}$$

Setting the 0 point of gravitational potential at the bottom of the loop, the gravitation potential of the bead is:

$$U_G = mgR\left(1 - \cos\theta\right) \tag{40}$$

Thus our Lagrangian is:

$$L = T - U = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - mgR\left(1 - \cos\theta\right) \tag{41}$$

Using:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \tag{42}$$

we can solve this

$$\frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta \tag{43}$$

and

$$\frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta \tag{44}$$

which leads to :

$$mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta = mR^2\ddot\theta \tag{45}$$

Clean up to make an equation of motion

$$\ddot\theta = \left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta \tag{46}$$

Now, what to do with this. We can't solve it analytically with elementary functions to get a general $\theta(t)$ equation. But, we can pick it apart a bit.

Equilibrium points (i.e. if you put the bead there, it stays there) can be found when

$$\ddot\theta = 0 \tag{47}$$

Thus:

$$\left(\omega^2\cos\theta - \frac{g}{R}\right)\sin\theta = 0 \tag{48}$$

So, if $\theta = 0$ then $\sin\theta = 0$ and we have an equilibrium point (i.e. at the bottom of the loop). Like for $\theta = \pi$, the top of the loop.

Another case:

$$\cos\theta = \frac{g}{\omega^2 R} \tag{49}$$

or

$$\theta_0 = \pm\cos^{-1}\left(\frac{g}{\omega^2 R}\right) \tag{50}$$

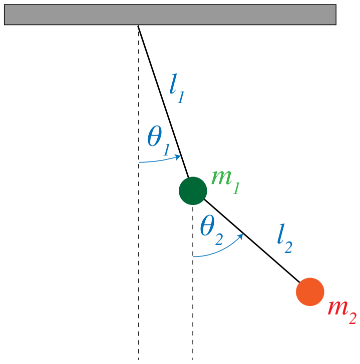
provided that $\omega^2 > g/R$

## 2.5 Double Pendulum

Each mass can have an $(x, y)$ coordinate given by:

$$(x,y)_1 = (l_1\sin\theta_1, -l_1\cos\theta_1) \tag{51}$$
$$(x,y)_2 = (l_1\sin\theta_1 + l_2\sin\theta_2, -l_1\cos\theta_1 - l_2\cos\theta_2) \tag{52}$$

Then we need to find the velocities (or $v^2$)




The Double Pendulum

$$v_1 = l_1^2\dot{\theta}_1^{\,2} \tag{53}$$
$$v_2 = l_1^2\dot{\theta}_1^{\,2} + l_2^2\dot{\theta}_2^{\,2} + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right) \tag{54}$$

The potential energies of each ball can be taken as:

$$U_1 = -m_1 g y_1 = -m_1 g l_1\cos\theta_1 \tag{55}$$
$$U_2 = -m_2 g y_2 = -m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2) \tag{56}$$

And therefore our Lagrangian:

$$L = T - U = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^{\,2} + \frac{1}{2}m_2 l_1^2\dot{\theta}_1^{\,2} + l_2^2\dot{\theta}_2^{\,2} + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)$$
$$+ m_1 g l_1\cos\theta_1 + m_2 g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

Next, do the partial derivatives for $\theta_1$ to obtain the equations of motion:

$$0 = (m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + m_2 l_1 l_2\dot{\theta}_2^{\,2}\sin(\theta_1 - \theta_2) + (m_1 + m_2)g l_1\sin\theta_1 \tag{57}$$

and $\theta_2$:

$$0 = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - m_2 l_1 l_2\dot{\theta}_1^{\,2}\sin(\theta_1 - \theta_2) + m_2 g l_2\sin\theta_2 \tag{58}$$

Simulation