Lagrangian Mechanics

1. Conservative vs. Non-conservative

1.1 The Lagrangian

$$L = T - U \tag{1}$$

This quantity is known as the **Lagrangian**. It is the difference between the kinetic and potential energies of a system.

Apply to Euler-Lagrange

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \tag{2}$$

leads to:

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x \tag{3}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = m\ddot{x} \tag{4}$$

so:

$$F_x = m\ddot{x} \tag{5}$$

Do it for all three dimensions and you have

$$-\nabla U = m\mathbf{a} \Rightarrow \mathbf{F} = m\mathbf{a} \tag{6}$$

2. Hamilton's Principle / Principle of Least Action

$$S[q_k(t)] = \int_{t_1}^{t_b} dt L(t, q_1, q_2, \ldots, \dot{q}_1, \dot{q}_2, \ldots) = \int_{t_a}^{t_b} L(t, q_k, \dot{q}_k) \tag{7}$$

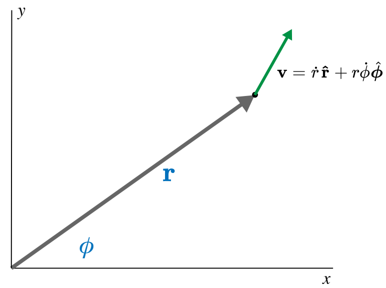
When S is stationary, i.e.

$$\delta S = \delta \int_{t_a}^{t_b} L(t, q_k, \dot{q}_k) = 0 \tag{8}$$

then the $q_k(t)$s will satisfy the equations of motions for the system between the boundary conditions.

2.2 Generalized Coordinates

Example Problem #1:



The basic polar coordinate system

Find the Lagrange Equations for a particle moving in two dimensions under the influence of a conservative force using polar coordinates

First find the Lagrangian: $L = T - U$

The kinetic energy will be given as usual by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) \tag{9}$$

and the potential can be written as:

$$U = U(r, \phi) \tag{10}$$

thus L is:

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r, \phi) \tag{11}$$

First consider **The r equation**

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \tag{12}$$

$$mr\dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m\dot{r}) = m\ddot{r} \tag{13}$$

The radial component of the force is just $-\partial U/\partial r$:

$$-\frac{\partial U}{\partial r} = F_r \tag{14}$$

Thus:

$$F_r = m\left(\ddot{r} - r\dot{\phi}^2\right) \tag{15}$$

which is recognizable as $F_r = ma_r$

Next, **The ϕ equation**

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} \tag{16}$$

Which leads to:

$$-\frac{\partial U}{\partial \phi} = \frac{d}{dt}\left(mr^2\dot{\phi}\right) \tag{17}$$

From vector calc:

$$-\frac{\partial U}{\partial \phi} = rF_\phi \quad (18)$$

which is the torque, τ and

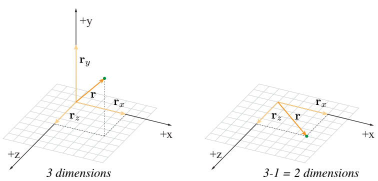
$$mr^2\dot{\phi} \quad (19)$$

is the angular momentum, L, thus, (17) can just be considered as

$$\tau = \frac{dL}{dt} \quad (20)$$

(where this L is angular momentum)

2.3 Force of Constraint



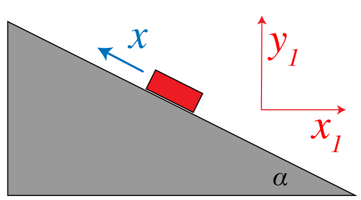
From 3 to 2 dimensions

By reducing the number of coordinates, we can imply a 'force of constraint'

Previously, we would model this as a particle subject to the force of gravity and a normal force. Now we can just say it exists in a 2d world, and we effectively accomplish the same thing.

Such a constraint is called **holonomic**.

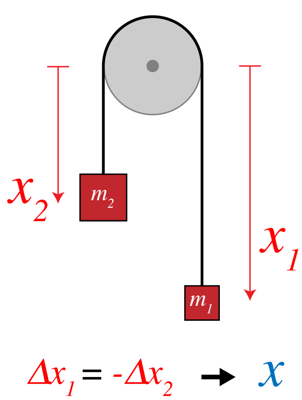
Examples of Constraints



A mass on a ramp

In this classic case, a naive application of Newton's Laws would suggest two coordinates. However, since the block is constrained to the surface of the ramp, there really is only one independent variable.

$$L = T - U = \frac{1}{2}mv^2 - U = \frac{1}{2}m\dot{x}^2 - mgx\sin\alpha \quad (21)$$



The Atwood Machine has 1 coordinate

Likewise, with the Atwood Machine, there really is only one coordinate.

Really, the string is a constant length, so

$$x_1 + x_2 + \pi R = l \ \text{ length of the string} \quad (22)$$

which means that:

$$x_2 = -x_1 + \text{const} \quad (23)$$

and also:

$$\dot{x_2} = -\dot{x_1} \quad (24)$$

So, letting $x_1 \to x$ Thus:

$$T = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2 \quad (25)$$

and the potential energy:

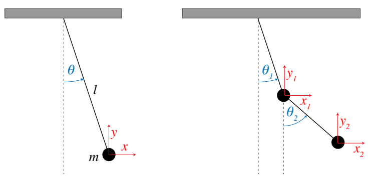
$$U = -m_1 g x_1 - m_2 g x_2 = -(m_1 - m_2)gx + \text{const} \quad (26)$$

Thus the Lagrangian is:

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2)gx \quad (27)$$

Solving for eq of motion:

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g \quad (28)$$

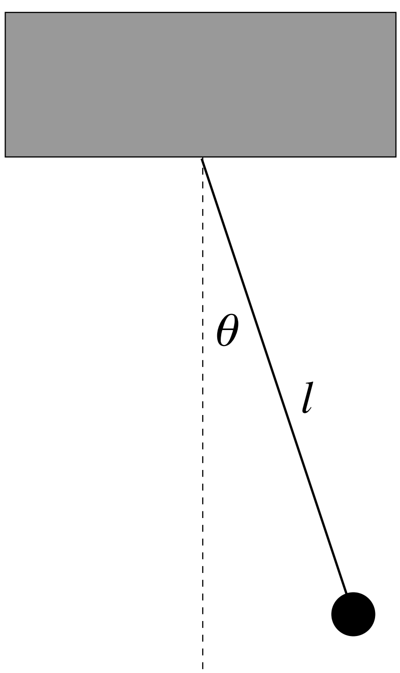


A regular pendulum and a double pendulum.

Pendulums are also great examples of systems with a constrain: the length of the string doesn't change.

3. Examples of Mechanical Systems:

Example Problem #2:



Pendulum on a stationary support

We'll start by finding the Lagrangian for the simple pendulum of length R and mass m. In cartesian coordinates, that would be:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \quad (29)$$

If we express that in polar coordinates:

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) \quad (30)$$

Two of these Three terms are zero, so we end up with:

$$T = \frac{1}{2}m\left(R^2\dot{\theta}^2\right) \quad (31)$$

The gravitational potential is

$$U = mgR(1 - \cos\theta) \quad (32)$$

Thus, our Lagrangian will be:

$$L = T - U = \frac{1}{2}m\left(R^2\dot{\theta}^2\right) - mgR(1 - \cos\theta) \quad (33)$$

Now we have singe *degree of freedom*, meaning there is only one coordinate, θ in this case. That makes solving the Euler-Lagrange Equation straightforward:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad (34)$$

Solving this:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\ddot{\theta} \tag{35}$$

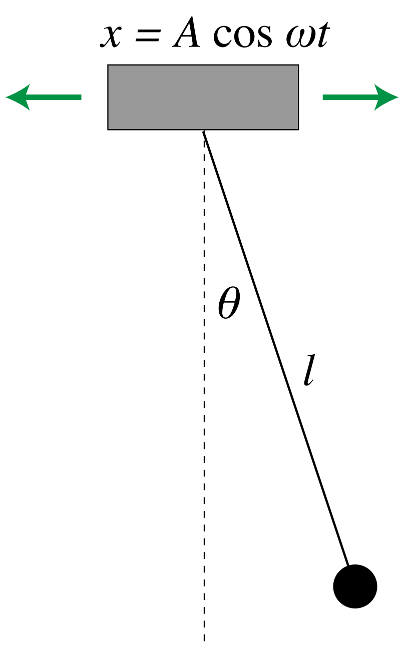
and

$$\frac{\partial L}{\partial \theta} = -mgR\sin\theta \tag{36}$$

Combining these two we have:

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0 \tag{37}$$

Example Problem #3:



Pendulum on Oscillating Support

To construct the Lagrangian, we need to find the speed of the mass. We can find the position of the mass and then take the time derivative:

$$\mathbf{r}_m = (r_x, r_y) = (x + l\sin\theta, -l\cos\theta) \tag{38}$$

Both x and θ have non-zero time derivatives, but l is a constant:

$$v^2 = v_x^2 + v_y^2 = \left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2 \tag{39}$$
$$= l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta \tag{40}$$

The potential energy can be found too: The gravitational potential is

$$U = -mgl\cos\theta \tag{41}$$

Thus the Lagrangian is:

$$L = T - U = \frac{1}{2}m\left(l^2\dot{\theta}^2 + \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) + mgl\cos\theta \tag{42}$$

Evaluate the Euler-Lagrange equation:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \tag{43}$$

$$\frac{d}{dt}\left(ml^2\dot{\theta} + ml\dot{x}\cos\theta\right) = -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta \tag{44}$$

which leads to:

$$l\ddot{\theta} + \ddot{x}\cos\theta = -g\sin\theta \tag{45}$$

We know from the setup that $x = A\cos\omega t$, so

$$l\ddot{\theta} - A\omega^2\cos(\omega t)\cos\theta + g\sin\theta = 0 \tag{46}$$

In the case of only small angles, this simplifies to:

$$\ddot{\theta} + \omega_0^2 \theta = a\omega^2 \cos(\omega t) \tag{47}$$

where $\omega_0 = \sqrt{g/l}$ and $a = A/l$

This is just a driven oscillator, that can now be solved using the standard methods.