Gravity and Orbits

[Sim from https://stellarium-web.org]

1. History of Planetary Motion

Claudius Ptolemy

Claudius Ptolemy (100 AD - 170 AD)

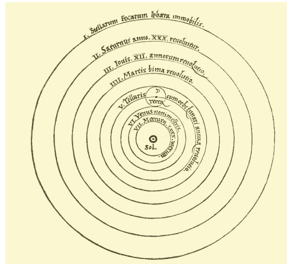
Epicycles Sim

1.1 Why circles?

Copernicus (1473-1543)

Copernicus was right: the earth and the other planets do orbit around the sun. However, he was still operating under the assumption that all the celestial bodies had to move in perfect circles. This turned out to be a rather major flaw in the framework.

Copernicus - woodcut attributed to Christoph Murer, from Nicolas Reusner's Icones (1587)

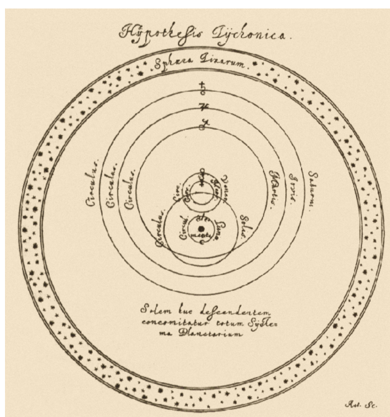


The Copernican System consisted of circular orbits centered around the sun.

Other astronomers of the era were unable to match the predictions of the Copernican system with observations. More 'tweaks' were added to the Copernican systems of circular heliocentric orbits in order to 'save the appearances'. In order to make the model match with observations, they still had to employ epicycles and so forth. However, the seed was planted. The sun should be at the center!

1.2 Tycho Brahe

Danish astronomer Tycho Brahe [1546-1601]. Made many very good measurements of the stars and planets.



The compromise.

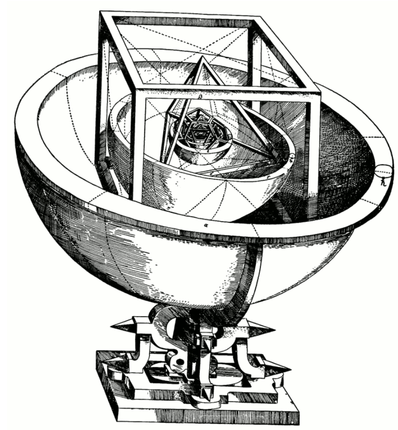
Late 16th century. Hybrid system: geoheliocentric model had the planets going around the sun, but the sun still went around the Earth. Close...

1.3 Johannus Kepler

Kepler (1571-1630)

Kepler played a major role in the 17th-century scientific revolution. He is best known for his laws of planetary motion, based on his works Astronomia nova, Harmonices Mundi, and Epitome of Copernican Astronomy. These works also provided one of the foundations for Newton's theory of universal gravitation.

German mathematician, astronomer, and astrologer. [1571-1630] Kepler was Tycho's assistant and was believed in the Copernican system.

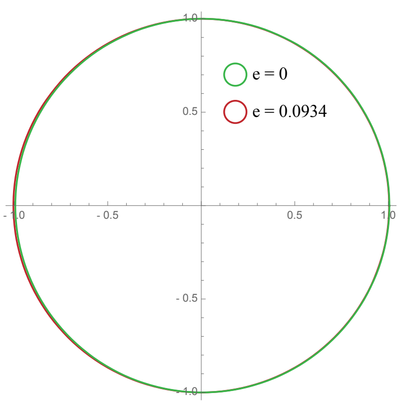


The Platonic Solid Model

2. Kepler's 3 laws of orbiting bodies

1. A planet orbits the sun in an ellipse. The Sun is at one focus of that ellipse.
2. A line connecting a planet to the Sun sweeps out equal areas in equal times
3. The square of a planet's orbital period is proportional to the cube of the average distance between the planet and the sun: $P^2 \propto a^3$.

2.1 1st Law: Ellipses

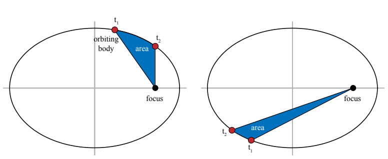


The orbit of Mars compared to a circle.

Mars' orbit [$e = 0.0934$] looks a lot like a circle.

ellipse-eccentricity Sim

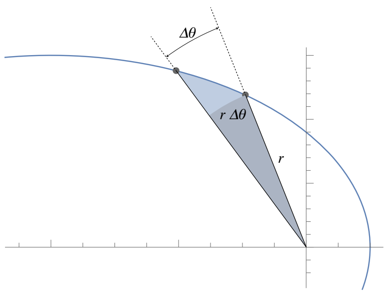
2.2 2nd Law: Equal Areas in Equal Times



Kepler's second law describes the areas swept out by an orbiting planet in a given time and says that for any given interval of time, the areas swept out will always be equal.

The blue areas in these figures will be the same if $t_2 - t_1$ is the same.

keplers-second-law Sim



An orbiting object moves through angle $\Delta\theta$ in a given time: Δt

If the Δt is very small, then the area swept out by the orbiter is:

$$dA = \frac{1}{2} r (r \Delta\theta) \tag{1}$$

Take the time derivative of this:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \tag{2}$$

but, our angular momentum for the orbiter is: $l = \mu r^2 \dot{\theta}$ so we can say:

$$\frac{dA}{dt} = \frac{\mu r^2 \dot{\theta}}{2\mu} = \frac{l}{2\mu} = \text{constant} \tag{3}$$

Integrating between two times:

$$A = \int_{t_1}^{t_2} \left(\frac{dA}{dt}\right) dt = \int_{t_1}^{t_2} \left(\frac{l}{2\mu}\right) dt = \left(\frac{l}{2\mu}\right)(t_2 - t_1) \tag{4}$$

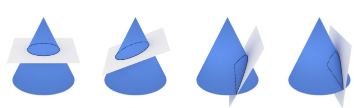
So, as long as the Δt is the same, then the area swept out will be equal. This is Kepler's 2nd law.

2.3 3rd Law

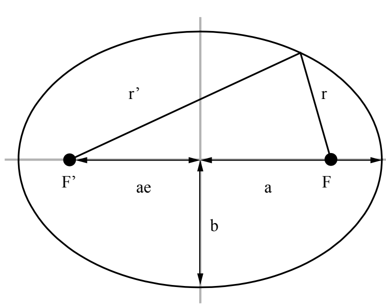
The square of a planet's orbital period is proportional to the cube of the average distance between the planet and the sun: $P^2 \propto a^3$.

$$t = \frac{2\pi a^{3/2}}{\sqrt{GM}} \tag{5}$$

3. Conic Sections



3.1 Ellipses



An ellipse satisfies this equation:

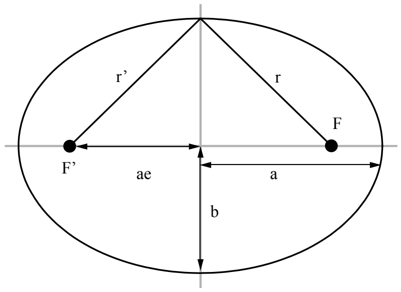
$$r + r' = 2a \tag{6}$$

- a is the **semi-major axis**
- r and r' are the distances to the ellipse from the two **focal points**, F and F'.
- e represents the eccentricity of the ellipse ($0 \leq e \leq 1$)

The **semi-major axis** is one-half the length of the long (i.e. major) axis of the ellipse.

Ellipse equation in Polar Coordinates

To Show:



Ellipse - the point where $r = r'$

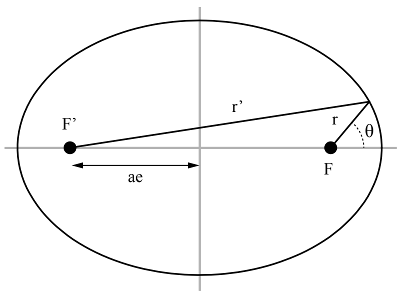
$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta} \tag{7}$$

We can start with basic ellipse equation Eq. (6) Take the point at the top of the semi-minor axis, where $r = r'$. Since $r + r' = 2a$ we know that $r = a$. Using the Pythagorean theorem, we can say that:

$$r^2 = b^2 + a^2e^2$$

Putting a in for r yields:

$$b^2 = a^2(1-e^2) \tag{8}$$



Ellipse - and polar coordinates r and θ.

For polar coordinates, we'll use r and θ, r being the distance from the principle focus and θ the angle measured counterclockwise from major axis of the ellipse. Again, from Pythagorus:

$$r'^2 = r^2\sin^2\theta + (2ae + r\cos\theta)^2$$

Expanding the 2nd RHS term:

$$\begin{aligned} r'^2 &= r^2\sin^2\theta + 4a^2e^2 + 2aer\cos\theta + r^2\cos^2\theta \\ &= r^2 + 4ae\left(ae + r\cos\theta\right) \end{aligned} \tag{9}$$

and using the fact that $r'^2 = (2a-r)^2$, we can obtain the polar equation for the ellipse shown above in eq (7).

$$r = \frac{a\left(1-\epsilon^2\right)}{1+\epsilon\cos\theta} \tag{10}$$

4. Characteristics of Different Orbits

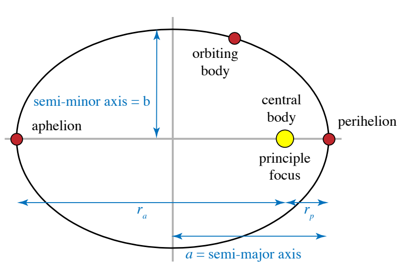
Circular:

The eccentricity $\epsilon = 0$, and the radius r is constant.

Both foci are located at the center.

These don't really exist in nature but are just mathematical possibilities

Elliptical



The basic parts of an elliptical orbit.

Are bound and closed orbits where the eccentricity is $0 < \epsilon < 1$

Have the gravitational center at the principle focus

$r_p = a(1-\epsilon)$ and $r_a = a(1+\epsilon)$

Shape is determined by any of the following pairs:

E	l
ϵ	r_p
a	b

Parabolic

$\epsilon = 1$

Like circles, these don't actually occur but are just mathematical possibilities.

Hyperbolic

$\epsilon > 1$

4.1 Bertrand's Theorem

The only central-force potentials $U(r)$ for which all bounded orbits are closed are:

1. Gravity: $U \propto \frac{1}{r}$
2. Springs: $U \propto r^2$

5. Two Bodies

binary-systems Sim

6. More than 2?