2d Motion

1. Trajectories of Projectiles

Before getting into a discussion of different coordinate systems, we'll look at a few basic examples of 2d kinematics. Sticking with Cartesian coordinates for now will be helpful conceptually, but we'll soon find situations where other coordinate systems are better choices.

[Sim from https://sciencesims.com/sims/launch-projectile/

1.1 Linear Drag - x & y components

Horizontal (i.e. no g)

$$m\ddot{x} = -b\dot{x}$$

leads to:

$$\begin{aligned} v_x(t) &= v_{x0} e^{-\frac{t}{\tau}} \\ x(t) &= v_{x0}\tau\left(1 - e^{-t/\tau}\right) \end{aligned} \tag{1}$$

where $\tau = m/b$

Vertical (i.e. w/ g)

$$m\ddot{y} = mg - b\dot{y}$$

This can be solved to obtain:

$$\begin{aligned} v_y(t) &= v_{\text{ter}} + (v_{y0} - v_{\text{ter}})e^{\frac{-t}{\tau}} \\ y(t) &= v_{\text{ter}}t + (v_{y0} - v_{\text{ter}})\tau\left(1 - e^{\frac{-t}{\tau}}\right) \end{aligned} \tag{2}$$

where $v_{\text{ter}} = mg/b$

$$m\ddot{y} = mg - b\dot{y}$$

When the two forces are equal, there is no more acceleration and we have reached terminal velocity: $v_{\text{ter}} = mg/b$.

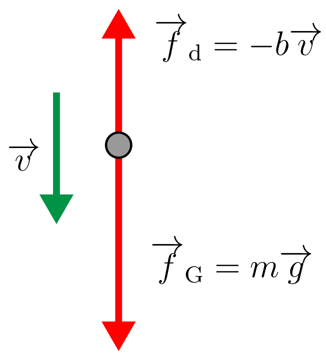
Our differential equation is then:

$$m\dot{v}_y = -b(v_y - v_{\text{ter}}) \tag{3}$$

Let $u = v_y - v_{\text{ter}}$ then we can write:

$$m\dot{u} = -bu$$

This familiar equation is solved with an exponential:



Free Body Diagram for Linear Drag

$$u = Ae^{-t/\tau}$$

with $\tau = m/b$

Replacing the u:

$$v_y - v_{\text{ter}} = Ae^{-t/\tau}$$

At $t = 0$, the velocity is v_{y0} so the constant can be found to be:

$$A = v_y - v_{\text{ter}}$$

And thus we obtain for the vertical velocity as a function of time:

$$v_y(t) = v_{\text{ter}} + (v_{y0} - v_{\text{ter}})\, e^{-t/\tau} \tag{4}$$

Since

$$y(t) = \int_0^t v_y(t) dt$$

obtaining the vertical position is done by performing the integral of (4).

$$y(t) = v_{\text{ter}} t + (v_{y0} - v_{\text{ter}})\tau \left(1 - e^{-t/\tau}\right) \tag{5}$$

Two dimensions - Two separated equations.

$$x(t) = v_{x0}\tau \left(1 - e^{-t/\tau}\right) \tag{6}$$

$$y(t) = (v_{y0} + v_{\text{ter}})\tau \left(1 - e^{\frac{-t}{\tau}}\right) - v_{\text{ter}} t \tag{7}$$

(note: for the following the direction of y is now positive-up.)

If we want to express $y(x)$, that is, eliminate time t from the equations, we can obtain:

$$y = \frac{v_{y0} + v_{\text{ter}}}{v_{x0}} x + v_{\text{ter}} \tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right) \tag{8}$$

Range - For Linear Drag

$$\frac{v_{y0} + v_{\text{ter}}}{v_{x0}} R + v_{\text{ter}} \tau \ln\left(1 - \frac{R}{v_{x0}\tau}\right) = 0 \tag{9}$$

How can we make sense of this? (Don't even bother trying to solve for R)

This is obtained by noting that in (8), when $y = 0$ we are either at the beginning or end of the trajectory. We expect there to be two solutions for $y = 0$, one with $x = 0$ as well, the other the final range of the motion: R.

Since $\tau = m/b$ and $v_{\text{ter}} = \mathrm{mg/b}$, we can expect both of these to be *large*. Thus $\frac{R}{v_{x0}\tau}$ from this term:

$$\ln\left(1 - \frac{R}{v_{x0}\tau}\right)$$

is *small*.

Using

$$\ln(1 - \epsilon) = -\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots\right) \tag{10}$$

We can arrive at:

$$\left[\frac{v_{y0}+v_{\text{ter}}}{v_{x0}}\right]R - v_{\text{ter}}\tau\left[\frac{R}{v_{x0}\tau}+\frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2+\frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3\right]=0 \tag{11}$$

This is a lot nicer to deal with. It can be rewritten as a 2nd order polynomial:

$$R=\frac{2v_{x0}v_{y0}}{g}-\frac{2}{3v_{x0}\tau}R^2 \tag{12}$$

(we've replaced $v_{\text{ter}}\tau$ with g)

If τ is large, i.e. little to no damping, then the R^2 term can be ignored, and we have the familiar range equation from physics in a vacuum.

$$R\approx\frac{2v_{x0}v_{y0}}{g}=R_{\text{vac}} \tag{13}$$

1.2 Quadratic Drag - x & y components

In cartesian coordinates:

$$\begin{aligned} m\ddot{\mathbf{r}} &= m\mathbf{g}-cv^2\ \hat{\mathbf{v}} && (14)\\ &= m\mathbf{g}-cv\ \mathbf{v} && (15)\end{aligned}$$

This, if you remember 2 kinematics from 20700, requires breaking the problem up into both directions. Previously, those two directions where independent, meaning what happens in one, stays there and doesn't affect the other. But, know, with the velocity dependent drag, they are *coupled*:

The two differential equations for a projectile with quadratic drag

$$m\dot{v}_x=-c\sqrt{v_x^2+v_y^2}\ v_x \tag{16}$$

$$m\dot{v}_y=-mg-c\sqrt{v_x^2+v_y^2}\ v_y \tag{17}$$

This is an example (already!) of something that has no analytic solution. That means we can't use separation of variable or any other regular diff.eq techniques to find a general solution to this. Numerical methods are the only way.