Coupled Oscillators

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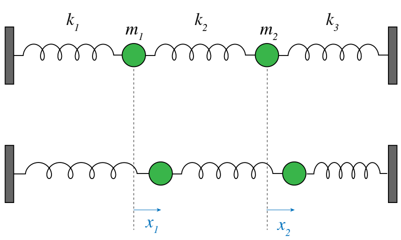
1. Connected Masses ($m(2)$ and $k(3)$)



Two masses, m_1 and m_2 connected via three springs.

Two masses can move horizontally. They are connected to each other and the walls by 3 springs as shown.

consider the forces on mass 1:

$$\begin{aligned} F_{\text{on mass 1}} &= -k_1 x_1 + k_2 (x_2 - x_1) \\ &= -(k_1 + k_2) x_1 + k_2 x_2 \end{aligned}$$

consider the forces on mass 2:

$$\begin{aligned} F_{\text{on mass 2}} &= -k_3 x_2 - k_2 (x_2 - x_1) \\ &= k_2 x_1 - (k_2 + k_3) x_2 \end{aligned}$$

This leads to 2 eqs. of motion.

$$\begin{aligned} m_1 \ddot{x}_1 &= -(k_1 + k_2) x_1 + k_2 x_2 \\ m_2 \ddot{x}_2 &= k_2 x_1 - (k_2 + k_3) x_2 \end{aligned}$$

1.1 Matrix Form

With the following definitions:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{2}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{3}$$

We can write:

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} \tag{4}$$

How to solve this?

Assume a complex solution:

$$\begin{aligned} z_1 &= x_1(t) + iy_1(t) = \alpha_1 e^{i(\omega t - \delta_1)} \\ &= \alpha_1 e^{-i\delta_1} e^{i\omega t} = a_1 e^{i\omega t} \end{aligned}$$

$$z_2 = \ldots = a_2 e^{i\omega t}$$

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t} = \mathbf{a} e^{i\omega t} \tag{5}$$

note: $\mathbf{x}(t) = \mathrm{Re}\ \mathbf{z}(t)$

Start with the equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} \tag{6}$$

Replace $\mathbf{x}$ with our complex $\mathbf{z}$ and we obtain:

$$-\omega^2 \mathbf{M}\ \mathbf{a}\ e^{i\omega t} = -\mathbf{K}\ \mathbf{a}\ e^{i\omega t} \tag{7}$$

Canceling the exponential terms and re-arranging:

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{a} = 0 \tag{8}$$

Now we ask, what do solutions to this equation look like?

Clearly, $\mathbf{a}$ could be zero, then we have the so-called *trivial solution*, i.e. nothing moves. Boring.

But if :

$$\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = 0 \tag{9}$$

then we might have more interesting solutions.

2. Equal m and k

Let's try a simple case, where the two masses are equal and the spring constants are all the same:

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \text{ and } \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \tag{10}$$

Our Matrix is thus:

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \tag{11}$$

Taking the determinant of this yields:

$$\det\left(\mathbf{K} - \omega^2 \mathbf{M}\right) = \left(2k - m\omega^2\right)^2 - k^2 \tag{12}$$

$$= \left(k - m\omega^2\right)\left(3k - m\omega^2\right) \tag{13}$$

There will be two frequencies that create a 0

$$\left(k - m\omega^2\right)\left(3k - m\omega^2\right) \tag{14}$$

$$\omega = \sqrt{\frac{k}{m}} = \omega_1 \text{ and } \omega = \sqrt{\frac{3k}{m}} = \omega_2 \tag{15}$$

Thus we have two non-trivial solutions. Our next step is to find the motions that occur at these frequencies

2.2 Normal modes

3. First Normal Mode

$$\left(\mathbf{K}-\omega^2\mathbf{M}\right)\mathbf{a}=0 \tag{16}$$

Start with $\omega_1=\sqrt{k/m}$

$$\left(\mathbf{K}-\omega_1^2\mathbf{M}\right)=\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=0 \tag{18}$$

which leads to two equations:

$$\begin{aligned} a_1-a_2&=0 \\ -a_1+a_2&=0 \end{aligned}$$

that both imply $a_1=a_2$

if we say:

$$a_1=a_2=Ae^{-i\delta}$$

then we can write our complex solution $\mathbf{z}(t)$ as:

$$\mathbf{z}(t)=\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}e^{i\omega_1 t}=\begin{bmatrix} A \\ A \end{bmatrix}e^{i(\omega_1 t-\delta)} \tag{19}$$

The actual motion of the masses is then described by the *real* part of the above:

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\begin{bmatrix} A \\ A \end{bmatrix}\cos\left(\omega_1 t-\delta\right) \tag{20}$$

$$\begin{aligned} x_1(t)&=A\cos(\omega_1 t-\delta) \\ x_2(t)&=A\cos(\omega_1 t-\delta) \end{aligned}$$

4. Second Normal Mode

Now Choose $\omega_2=\sqrt{3k/m}$

$$\left(\mathbf{K}-\omega_2^2\mathbf{M}\right)=\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=0 \tag{22}$$

which leads to two equations:

$$a_1+a_2=0$$

which implies $a_1=-a_2$

if we say:

$$a_1=-a_2=Ae^{-i\delta}$$

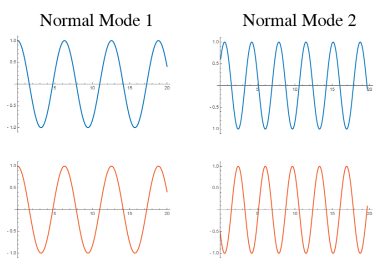
then we can write our complex solution $\mathbf{z}(t)$ as:

$$\mathbf{z}(t) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega_2 t} = \begin{bmatrix} A \\ -A \end{bmatrix} e^{i(\omega_2 t - \delta)} \tag{23}$$

The motion of the masses is then described by the *real* part of the above:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A \\ -A \end{bmatrix} \cos\left(\omega_2 t - \delta\right) \tag{24}$$

$$x_1(t) = A\cos(\omega_2 t - \delta)$$
$$x_2(t) = -A\cos(\omega_2 t - \delta)$$



Two Normal Modes

4.3 General Solution

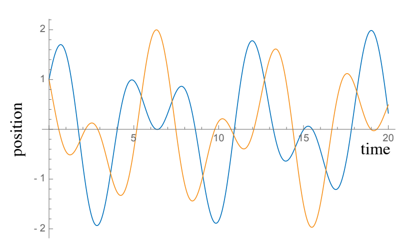
$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos\left(\omega_1 t - \delta_1\right)$$
$$\mathbf{x}(t) = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos\left(\omega_2 t - \delta_2\right)$$

These are both solutions to

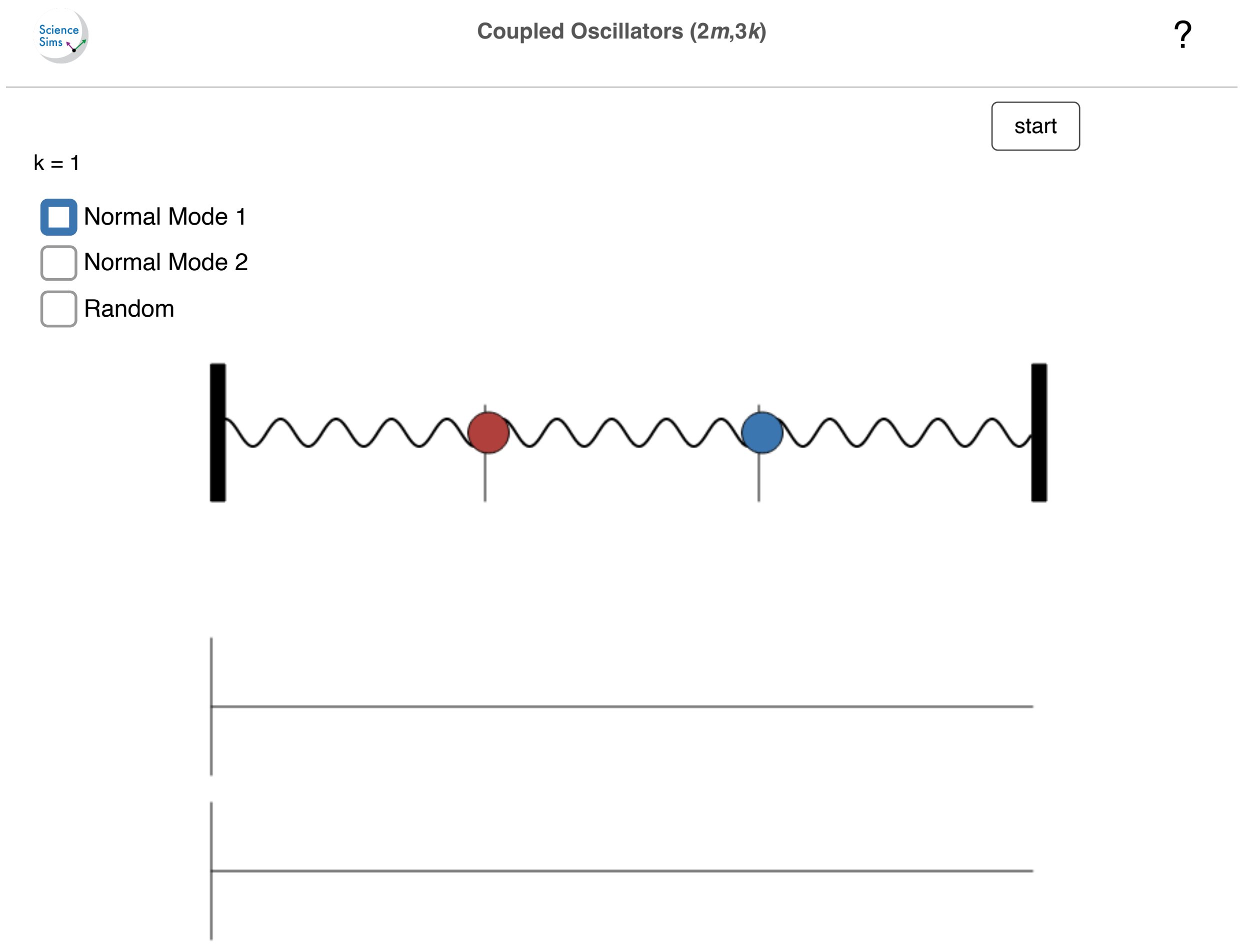
$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x} \tag{25}$$

which means their sum is also also a solution.

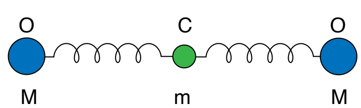
$$\mathbf{x}(t) = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos\left(\omega_1 t - \delta_1\right) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos\left(\omega_2 t - \delta_2\right) \tag{26}$$



An arbitrary plot for the general solution.

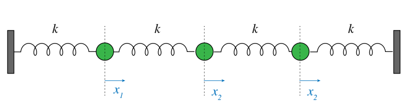


4.4 Physical Example



A Carbon Dioxide Molecule

5. Larger systems



More masses?

Now, let's try it with the Lagrangian formulation:

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \tag{27}$$

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}kx_3^2 \tag{28}$$

Thus our L is:

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_2^2) - \frac{1}{2}kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2 - \frac{1}{2}kx_3^2 \tag{29}$$

After solving the Euler-Lagrange for this system:

$$
\begin{aligned}
m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\
m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) \\
m\ddot{x}_3 &= -kx_3 - k(x_3 - x_2)
\end{aligned}
$$

Adopt exponential solutions:

$$
\begin{aligned}
x_1 &= b_1 e^{i\omega t} \\
x_2 &= b_2 e^{i\omega t} \\
x_3 &= b_3 e^{i\omega t}
\end{aligned}
$$

Which leads to a 3 x 3 matrix version:

$$
\begin{pmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0 \tag{30}
$$

Taking the determinant of this and setting equal to zero:

$$
\begin{vmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{vmatrix} = 0 \tag{31}
$$

Will lead to a polynomial

$$
(-m\omega^2 + 2k)[(-m\omega^2 + 2k)^2 - k^2] + k(-k(-m\omega^2 + 2k)) = 0 \tag{32}
$$

This can be factored:

$$
(-m\omega^2 + 2k)[(-m\omega^2 + 2k)^2 - 2k^2] = 0 \tag{33}
$$

if

$$
(-m\omega^2 + 2k) = 0 \tag{34}
$$

then we have a root:

$$
\omega_1 = \sqrt{\frac{2k}{m}} \tag{35}
$$

likewise: if

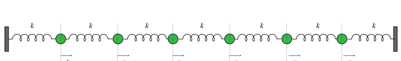
$$
[(-m\omega^2 + 2k)^2 - 2k^2] = 0 \tag{36}
$$

then:

$$
\omega_2 = \sqrt{\frac{(2 - \sqrt{2})k}{m}} \text{ and } \omega_3 = \sqrt{\frac{(2 + \sqrt{2})k}{m}} \tag{37}
$$

$$
\begin{aligned}
b_2 &= 0, & b_3 &= -b_1 & &\text{for } \omega = \omega_1 \\
b_3 &= b_1, & b_2 &= \sqrt{2} b_1 & &\text{for } \omega = \omega_2 \\
b_3 &= b_1, & b_2 &= -\sqrt{2} b_1 & &\text{for } \omega = \omega_3
\end{aligned}
$$

6. The continuum limit



$$
L = T - U = \sum_i \frac{1}{2} m\dot{x}_i^2 - \frac{1}{2}\sum_i k(x_{i+1} - x_i)^2 \tag{38}
$$

The lagrangian for a mass m is then:

$$m\ddot{x}_i - k(x_{i+1} - x_i) + k(x_i - x_{i-1}) = 0 \tag{39}$$

or:

$$k(x_{i+1} - x_i) - k(x_i - x_{i-1}) = m\ddot{x}_i \tag{40}$$

Define a mass per unit length:

$$\mu = \frac{m}{a} \tag{41}$$

as well as the Young's Modulus:

$$Y = \frac{\text{stress}}{\text{strain}} \tag{42}$$

$$\text{stress} = T = k(x_{i+1} - x) \tag{43}$$

and

$$\text{strain} = \frac{x_{i+1} - x_i}{a} \tag{44}$$

Stress: is tension / Strain is extension per unit length

Thus:

$$Y = \frac{k(x_{i+1} - x_i)}{(x_{i+1} - x_i)/a} = ka \tag{45}$$

Now, let $a \to 0$

$$\mu\ddot{x}_i - \frac{Y}{a}[(x_{i+1} - x_i) - (x_i - x_{i-1})] = 0 \tag{46}$$

Call $\eta(x,t)$ the displacement from equilibrium

$$\ddot{x}_i \to \left.\frac{\partial^2\eta}{\partial t^2}\right|_{x_i} \tag{47}$$

$$\frac{(x_{i+1} - x_i)}{a} \to \left.\frac{\partial\eta}{\partial x}\right|_{x_i+a} \tag{48}$$

$$\frac{(x_i - x_{i-1})}{a} \to \left.\frac{\partial\eta}{\partial x}\right|_{x_i} \tag{49}$$

From Taylor Series:

$$\left.\frac{\partial\eta}{\partial x}\right|_{x_i+a} = \left.\frac{\partial\eta}{\partial x}\right|_{x_i} + a\left.\frac{\partial^2\eta}{\partial x^2}\right|_{x_i+a} \tag{50}$$

6.5 Wave Equation:

$$\frac{\partial^2\eta}{\partial x^2} - \frac{\mu}{Y}\frac{\partial^2\eta}{\partial t^2} = 0 \tag{51}$$

Replace the μ/Y with the velocity, v^2.

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \eta}{\partial t^2} = 0 \tag{52}$$