Conservation Laws

1. Linear Momentum

$$\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}} \tag{1}$$

$$\mathbf{P} = \sum m_\alpha \mathbf{v}_\alpha \tag{2}$$

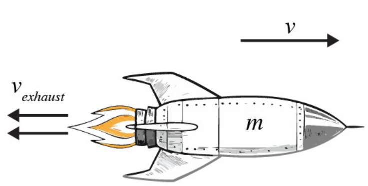
The change of a system's linear momentum is determined by the external forces acting on the system.

[Sim from https://sciencesims.com/sims/simple-collision]

[Sim from https://sciencesims.com/sims/uneven-ellastic-collision]

[Sim from https://sciencesims.com/sims/2d-elastic-collision]

1.1 Variable Mass Systems



A rocket with mass m moves to the right at speed v. The exhaust moves to the left at speed $v_{exhaust}$

$$P(t) = mv \tag{3}$$

$$\begin{aligned} P(t+dt) &= (m+dm)(v+dv) - dm(v - v_{ex}) \quad (4)\\ &= mv + m\,dv + dm\,v_{ex} \quad (5)\end{aligned}$$

$$dP = m\,dv + dm\,v_{ex} \tag{6}$$

If no external forces are present, then $\dot{P} = 0$ and we have

$$m\,dv = -dm\,v_{ex} \tag{7}$$

or, after dividing by dt

$$m\dot{v} = -\dot{m}v_{ex} \tag{8}$$

But, $m\dot{v}$ is just $m\,a$ so we can call the R.H.S of (8) the **thrust**

$$\text{thrust} = -\dot{m}v_{ex} \tag{9}$$

> Example Problem #1:
>
> Solve

> $$m\,dv = -dm\,v_{ex}$$
>
> for $v(t)$. (Initial speed and mass are v_0 and m_0 respectively.)

2. Angular Momentum

2.2 For a single particle:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{10}$$

$$\dot{\mathbf{L}} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \tag{11}$$

$$= (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times \dot{\mathbf{p}}) \tag{12}$$

but, $\dot{\mathbf{r}} \times \mathbf{p}$ = 0 (why?) and $\dot{\mathbf{p}} = \mathbf{F}$

Thus we have the torque, $\boldsymbol{\tau}$, as the change in angular momentum

$$\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F} \equiv \boldsymbol{\tau} \tag{13}$$

3. Energy

3.3 Work & Energy

$$T = \frac{1}{2}mv^2 \tag{14}$$

How does the kinetic energy change in time?

$$\frac{dT}{dt} = \frac{1}{2}m\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) \tag{15}$$

$$= \frac{1}{2}m\left(\dot{\mathbf{v}} \cdot \mathbf{v} + \mathbf{v} \cdot \dot{\mathbf{v}}\right) \tag{16}$$

$$= m\dot{\mathbf{v}} \cdot \mathbf{v} \tag{17}$$

But, $m\ \dot{\mathbf{v}}$ is the same as the force, $\mathbf{F}$ (2nd law), so:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{18}$$

or,

$$dT = \mathbf{F} \cdot d\mathbf{r} \tag{19}$$

Using an integral formulation lets use express the change in kinetic energy for any arbitrary path:

$$\Delta T = T_2 - T_1 = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \equiv W(1 \to 2) \tag{20}$$

3.4 Conservative Forces

1. Force depends only on position $\mathbf{r}$
2. And is path independent

If we have a conservative force, then we can also talk about the potential energy associated with that force.

$$U(r) = -W(\mathbf{r}_0 \to \mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \tag{21}$$

Thus we can arrive at a statement for the Conservation of Mechanical Energy, E. Since:

$$\Delta T = -\Delta U \tag{22}$$

$$\Delta \left(T + U \right) = 0 \tag{23}$$

and then:

$$E = T + U \tag{24}$$

If there are non-conservative forces present, i.e. friction, then the change in mechanical energy is just the work done by the non-conservative force:

$$\Delta E = \Delta \left(T + U \right) = W_{\mathrm{nc}} \tag{25}$$

$$\mathbf{F} = -\nabla U \tag{26}$$