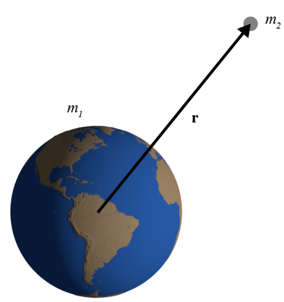
Central Forces

1. Central Forces

Spherically symmetric forces



Earth and a 2nd mass.

Gravity is a *central force*

It acts between 2 objects of masses: m_1 and m_2 and depends on their distance: r.

$$\mathbf{F} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} \tag{1}$$

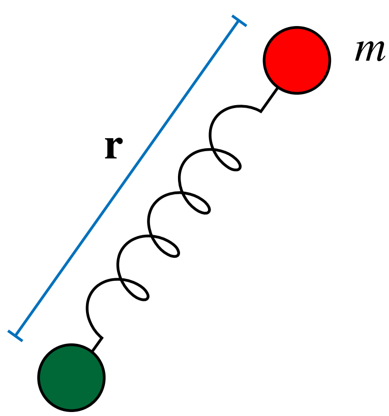
The unit vector $\hat{\mathbf{r}}$ points away from the source object.

The gravitational potential is easily found by considering the integral of F over a distance r.

$$U(r) = -\int F(r)dr = -G\frac{m_1 m_2}{r} \tag{2}$$

Springs (or spring-like)

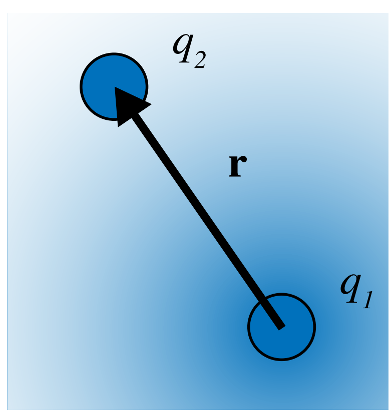
$$\mathbf{F} = -k\mathbf{r} = -kr\hat{\mathbf{r}} \tag{3}$$



Springs

Coulomb Force

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}} \tag{4}$$

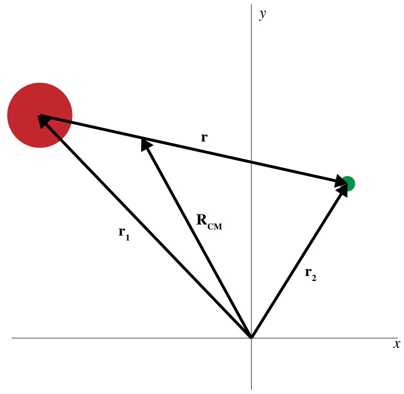


The Coulomb Force

However, we usually assumed the *source* didn't also move, or if it did, it only moved a little bit and we could ignore it.

Obviously, if we are considering the gravitational interaction between two objects with similar masses, then we can't assume this to be the case.

2. The Two-Body Problem



Two bodies - Center of Mass

and

$$\mathbf{R}_{\text{cm}} \equiv \frac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1+m_2} \tag{5}$$

$$\mathbf{r} \equiv \mathbf{r_2} - \mathbf{r_1} \tag{6}$$

Solving for $\mathbf{r_1}$ and $\mathbf{r_2}$ in terms of $\mathbf{R}_{\text{cm}}$ and $\mathbf{r}$

$$\mathbf{r_1} = \mathbf{R}_{\text{cm}} - \frac{m_2}{M}\mathbf{r} \tag{7}$$
$$\mathbf{r_2} = \mathbf{R}_{\text{cm}} + \frac{m_1}{M}\mathbf{r} \tag{8}$$

Using those two coordinates, we can express the kinetic energy of each particle:

$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 \tag{9}$$

After some mildly tedious algebra, this becomes:

$$T = \frac{1}{2}M\dot{\mathbf{R}}_{\text{cm}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 \tag{10}$$

where $\mu = \frac{m_1m_2}{m_1+m_2} = \frac{m_1m_2}{M}$ is the *reduced mass* for the two-body system.

[Sim from https://sciencesims.com/sims/two-body-COM/]

This kinetic energy can be conceptualized as just the kinetic energies of two particles. (Neither of which really exist)

$$T = \underbrace{\frac{1}{2}M\dot{\mathbf{R}}_{\text{cm}}^2}_{\text{Particle 1}} + \underbrace{\frac{1}{2}\mu\dot{\mathbf{r}}^2}_{\text{Particle 2}} \tag{11}$$

These ficticious particles can be interpreted as

1. A particle of mass M moving with the speed of the center of mass, $\dot{R}_{\text{cm}}$
2. A particle of mass μ moving with the speed of the relative position, $\dot{r}$

Let's construct a Lagrangian:

$$L = T - U = \frac{1}{2}M\dot{\mathbf{R}}_{\text{cm}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r) \tag{12}$$

Looking at this, we can see right away that one of the coordinates is cyclic (or ignorable): $\mathbf{R}_{\text{cm}}$ doesn't appear in the Lagrangian so we know that:

$$\mathbf{P} = M\dot{\mathbf{R}}_{\text{cm}} \tag{13}$$

is conserved. (That's the total momentum of the system)

$$L = \underbrace{\frac{1}{2}M\dot{\mathbf{R}}_{\text{cm}}^2}_{\text{cm}} + \underbrace{\frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r)}_{\text{rel. position}} \tag{14}$$

$$L = L_{\text{cm}} + L_{\text{relative position}} \tag{15}$$

Re-writing the second part of the Lagrangian in polar coordinates (and realizing that this is effectively now a 2d system):

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) \tag{16}$$

Two things to note:

1. L is not explicitly *time dependent*, which in this case means the total energy ($T + U$) is conserved.
2. The generalized coordinate ϕ is *cyclic*, so that means that

$$p_\phi \equiv l = \mu r^2\dot{\phi} = r(\mu r\dot{\phi}) = \textbf{constant} \tag{17}$$

The achievement?

We've dealt with a non-inertial frame (i.e. the accelerative motion of the source) by describing the motion of a particle of mass μ around the source.

3. Effective Potential

Two conservation equations:

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) \tag{18}$$

and

$$l = \mu r^2\dot{\phi} \tag{19}$$

To solve these analytically, we have to eliminate either t or ϕ

We can try to obtain two different descriptions:

$$r(t) \;\; \text{or} \;\; r(\phi) \tag{20}$$

Start with getting rid of ϕ

Angular momentum conservation let's us rewrite $\dot{\phi}$ in terms of r:

$$\dot{\phi} = \frac{l}{mr^2} \tag{21}$$

Which we can then use to recast the Energy as:

$$\begin{aligned} E &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + U(r) \\ &= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\left(\frac{l}{\mu r^2}\right)^2\right) + U(r) \end{aligned}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}} \tag{22}$$

with

$$U_{\text{eff}} = \frac{l^2}{2\mu r^2} + U(r) \tag{23}$$

The term: $\frac{l^2}{2\mu r^2}$ is the **centrifugal potential**

Now we consider the first derivative of U_{eff} w.r.t r.

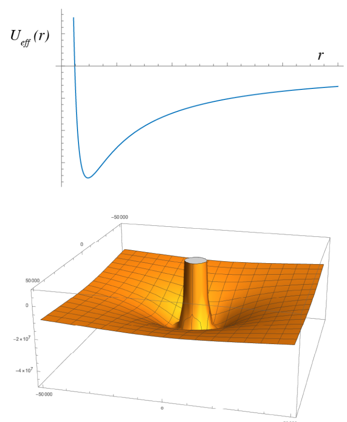
$$U'_{\text{eff}}\big|_{r=R} = -\frac{l^2}{\mu r^3} + U'(r) = 0 \tag{24}$$

For the case of gravity:

$$U_G(r) = -\frac{Gm_1 m_2}{r} \tag{25}$$

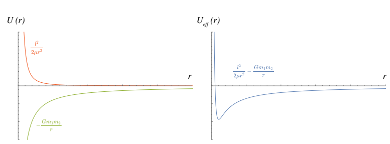
and so our effective potential is:

$$U_{\text{eff}} = \frac{l^2}{2\mu r^2} - \frac{Gm_1 m_2}{r} \tag{26}$$



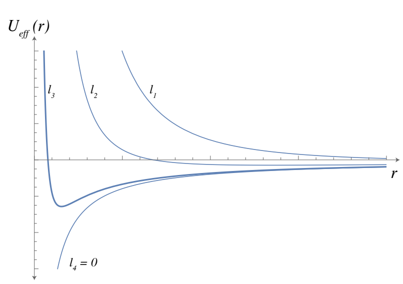
Effective Potential Gravity

The U_{eff} is comprised of two contributions: one from the gravitational potential, and one from the 'centrifugal' potential.

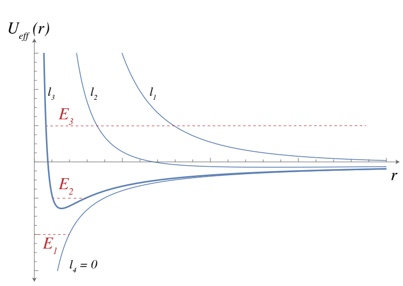


Two Contributions to U_{eff}

Different l values



Different E for an orbiter



[Sim from https://sciencesims.com/sims/orbit-ang-momentum-conics/]

[Sim from https://sciencesims.com/sims/effective-potential-gravity/]

Find $r(t)$ or $t(r)$

$$\frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) = E \tag{27}$$

with

$$U_{\text{eff}} = \frac{l^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \tag{28}$$

Solve for $\dot{r}$...

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E + \frac{GM\mu}{r} - \frac{l^2}{2\mu r^2}\right)} \tag{29}$$

$$t(r) = \pm\sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{r\,dr}{\sqrt{Er^2 + GM\mu r - \frac{l^2}{2\mu}}} \tag{30}$$

Let:

$$X = cx^2 + bx + a = Er^2 + GM\mu r - \frac{l^2}{2\mu} \tag{31}$$

Now, from a table of integrals:

$$\int\frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int\frac{dx}{\sqrt{X}} \tag{32}$$

and

$$\int\frac{dx}{\sqrt{x}} = \frac{-1}{\sqrt{-c}}\sin^{-1}\left(\frac{2cx+b}{\sqrt{-\left(4ac-b^2\right)}}\right) \tag{33}$$

Time from periapse to apoapse (closest to farthest) is:

$$t(p\rightarrow a) = GM\left(\frac{\mu}{2(-E)}\right)^{3/2}\pi \tag{34}$$

Now eliminate t to get $r(\phi)$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + U(r) \tag{35}$$

and

$$l = mr^2\dot{\phi} \tag{36}$$

$$\frac{dr}{d\phi} = \frac{dr/dt}{d\phi/dt} \tag{37}$$

$$\frac{dr}{d\phi} = \pm\sqrt{\frac{2m}{l^2}}r^2\sqrt{E - \frac{l^2}{2mr^2} - U(r)} \tag{38}$$

$$\phi = \int d\phi = \pm\frac{l}{\sqrt{2m}}\int^r \frac{r^{-2}}{\sqrt{E - \frac{l^2}{2mr^2} - U(r)}}dr \tag{39}$$

For Gravitational Forces:

$$U(r) = -\frac{GMm}{r} \tag{40}$$

This leads to a similar integral:

$$\phi = \int d\phi = \pm\frac{l}{\sqrt{2m}}\int \frac{dr}{r\sqrt{Er^2 + GMmr - \frac{l^2}{2m}}} \tag{41}$$

From the integral tables:

$$\int \frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bx + 2a}{x\sqrt{-(4ac - b^2)}}\right) \tag{42}$$

Carry this through as before, and define:

$$\epsilon \equiv \sqrt{1 + \frac{2El^2}{G^2M^2m^3}} \tag{43}$$

after some algebra:

$$r = \frac{l^2/GMm^2}{1 + \epsilon\cos(\phi)} \tag{44}$$

with

$$\epsilon \equiv \sqrt{1 + \frac{2El^2}{G^2M^2m^3}} \tag{45}$$