Calculus of Variations

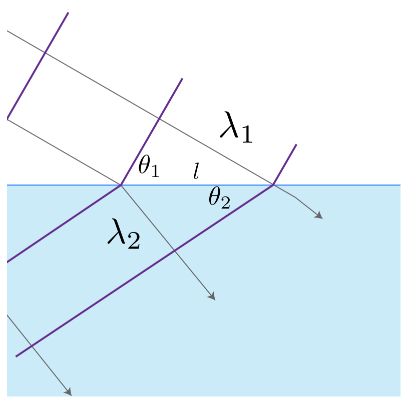
1. Origins

1.1 Snell's Law

Snell's Law, as we know it today, describes the path a light ray takes as it crosses a boundary between two regions of space with different materials. The materials may have different index of refraction values, which leads to difference speeds for light waves, which in turn leads to the phenomenon we call refraction.

Snell's Law, as normally written, is just a prescription of what to expect. It doesn't contain an causal significance. In other words, you can't take the law as is and say really why it's true. It describes the phenomena, but doesn't give a lot of hints as to why light behaves this way.

[Sim from https://sciencesims.com/sims/snells-law]

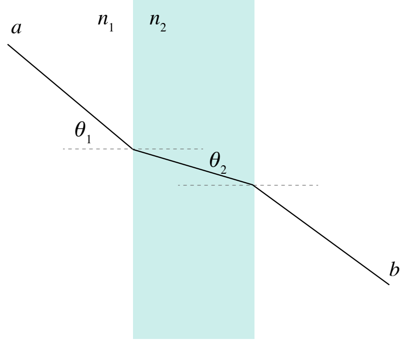


Waves at an interface

If one treats light as a wave-like phenomena, then Snell's law is readily derivable from geometry.

But, back in 1600's the nature of light (i.e. particle or wave) was anything but settled.

1.1 Principle of Least Time



Q: How does a photon (i.e. not a wave) decide to change directions upon entering and exiting the medium?

A: It has to get to point b in the shortest time.

Light traveling between *a* and *b*. How does it know which way to go?

Since the distance s traveled in time t at a speed v is

$$v = \frac{s}{t} \tag{1}$$

and using the fact that the speed of light depends on the local index of refraction:

$$v_{\text{light}} = \frac{c}{n(\mathbf{r})} \tag{2}$$

we can express the time to travel an infinitesimal distance ds as :

$$dt = \frac{ds}{v} = \frac{ds}{c/n(\mathbf{r})} \tag{3}$$

Integrating this to find the total time it takes:

$$t = \int dt = \frac{1}{c}\int n(\mathbf{r})ds \tag{4}$$

Finding the path (ds) that minimizes the total time is essentially the **Principle of Least Time**, though it would take a few more hundred years to figure out how to really do this math. (and even longer to figure out why it should even be true)

Assuming the index of refraction in our material $n(\mathbf{r})$ only changes with respect to the x position, we could write:

$$t = \frac{1}{c}\int ds\; n(x) \tag{5}$$

or, using i.47

$$t = \frac{1}{c}\int n(x)\sqrt{dx^2 + dy^2} = \frac{1}{c}\int n\left(x\right)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx \tag{6}$$

which, using the shorthand for $dy/dx = y'$ (in general, a prime will mean take the derivative w.r.t whatever the explicit independent variable is),

$$t = \frac{1}{c}\int n(x)\sqrt{1 + y'^2}dx \tag{7}$$

Lastly, the index of refraction should not be limited to just x dependent changes: $n(x, y)$ is more general.

$$t = \frac{1}{c}\int n(x, y)\sqrt{1 + y'^2}dx = \int F(x, y(x), y'(x))dx \tag{8}$$

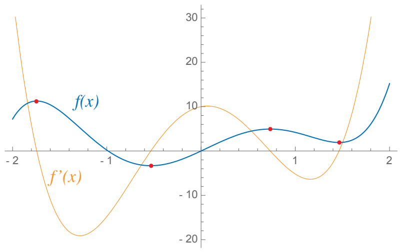
Now, the function F can be seen to depend on x, $y(x)$, and $y'(x)$. The techniques that follow will seek to find the path $y(x)$ that minimizes this integral.

Getting even more general, if:

$$I = \int F(x, y(x), y'(x))dx \tag{9}$$

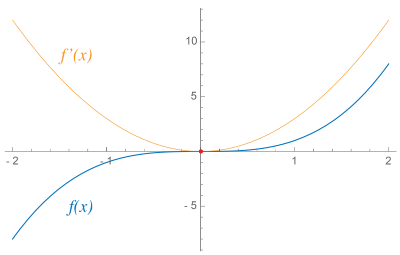
Can we find paths $y(x)$ that also **maximize** I? Or, even find what we'll call a **stationary** path, where the value of I is nearly independent of small changes in the path?

Review of Mins/Max/Saddles



The Minimums and Maximums of a 1d function.

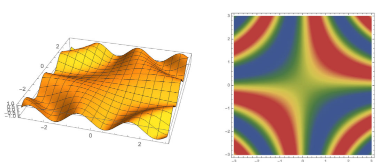
The mins & maxes of a function are just the zero-crossings of the first derivative.



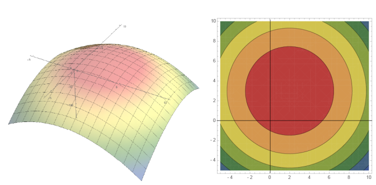
A saddle point in 1d

Saddle points can also exist.

Functions of Two Variables



Functions of 2 variables



An inverted paraboloid

Create this plot:

```
a = 1;
x0 = 2; y0 = 3;
paraboloid [x_, y_] := -a ((x - x0)^2 + (y - y0)^2)
Plot3D[paraboloid[x, y], {x, -5, 10}, {y, -5, 10}, Boxed ->
False,
  AxesOrigin -> {0, 0, 0}, PlotStyle -> Opacity[0.5],
  ColorFunction -> "DarkRainbow"]
ContourPlot[paraboloid[x, y], {x, -5, 10}, {y, -5, 10}, Frame ->
True,
   AxesOrigin -> {0, 0, 0}, ColorFunction -> "DarkRainbow"]
```

Example Problem #1:

Find the max of this inverted paraboloid

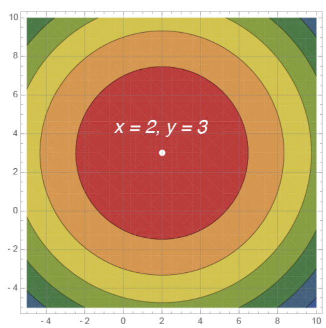
$$f(x,y) = -a\left((x-x_0)^2 + (y-y_0)^2\right) \tag{10}$$

Set both partials equal to zero:

$$\frac{\partial f}{\partial x} = -2a(x - x_0) = 0 \ , \ \frac{\partial f}{\partial y} = -2a(y - y_0) = 0$$

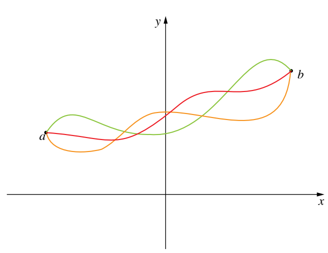
Or,

```
Solve[D[paraboloid[x, y] == 0, x], x]
Solve[D[paraboloid[x, y] == 0, y], y]
```




```
a = 1;
x0 = 2; y0 = 3;
paraboloid [x_, y_] := -a ((x - x0)^2 + (y - y0)^2)
Maximize[paraboloid[x, y], {x, y}]
Show[ContourPlot[paraboloid[x, y], {x, -5, 10}, {y, -5, 10},
  Frame -> True, AxesOrigin -> {0, 0, 0},
  ColorFunction -> "DarkRainbow", GridLines -> Automatic],
 Graphics[{White, PointSize[Large], Point[{x, y} /. Last[%]]}]]
```

The maximum of this paraboloid



The path length for all the different paths from a to b can be considered a function.

The **function** is now what we want to find stationary points for.

2. Examples

$$\frac{\partial F}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right) = 0 \tag{11}$$

Example Problem #2:

Show that the shortest distance between 2 points is a straight line

$$s = \int ds = \sqrt{dx^2 + dy^2} = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{12}$$

or, more simply:

$$s = \int_{x_a}^{x_b} \sqrt{1+y'^2}dx \tag{13}$$

Thus, our F is:

$$F = \sqrt{1+y'^2} \tag{14}$$

Since

$$\frac{\partial F}{dy} = 0$$

, we can use 11 to say that:

$$\frac{d}{dx}\left(\frac{\partial F}{dy'}\right) = 0 \tag{15}$$

which means that

$$\frac{\partial F}{\partial y'} = \text{A constant} = k \tag{16}$$

Next:

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} = k \tag{17}$$

Solve for y'

$$y' = \frac{\pm k}{\sqrt{1-k^2}} \equiv m_1 \tag{18}$$

Now, integrate dy/dx:

$$dy = \int m_1 dx = m_1 x + m_2 \tag{19}$$

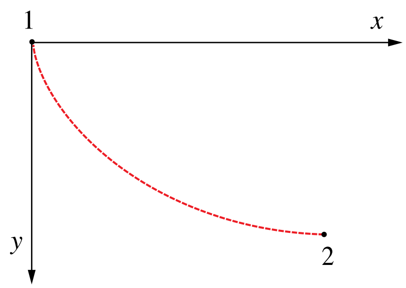
Example Problem
#3:

Find the shape of a ramp that will bring a sliding mass to a lower point the fastest.

The travel time for a small distance ds at a (varying) speed v is:

$$t = \int \frac{ds}{v} \tag{20}$$

$$ds = \sqrt{dx^2 + dy^2} \tag{21}$$



The geometry of the brachistochrone problem.

Using conservation of energy,

$$E = \frac{1}{2}mv^2 + mg(-y) = 0 \tag{22}$$

Thus, filling out our t:

$$t = \int \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx \tag{23}$$

We can also switch to x' and write:

$$t = \int \frac{\sqrt{1+x'^2}}{\sqrt{2gy}} dy \tag{24}$$

and use:

$$\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial x'} = 0 \tag{25}$$

which will make Euler-Lagrange easier to solve, since the $\partial F/\partial x$ will vanish.

Thus:

$$\frac{\partial F}{\partial x'} = \frac{1}{\sqrt{2gy}} \frac{x'}{\sqrt{1+x'^2}} = k \tag{26}$$

Solving for x'

$$x' = \frac{\pm k\sqrt{2gy}}{\sqrt{1-2k^2 gy}} \equiv \sqrt{\frac{y}{a-y}} \tag{27}$$

Let $a = 1/(2k^2 g)$ and integrate:

$$x = \int dx = \int dy \sqrt{\frac{y}{a-y}} \tag{28}$$

Solving using substitution of

$$y = a\sin^2\left(\frac{\theta}{2}\right) = \frac{a}{2}(1-\cos\theta) \tag{29}$$

we can obtain

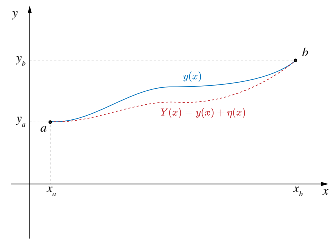
$$x = \frac{a}{2}(\theta - \sin\theta) \tag{30}$$
$$y = \frac{a}{2}(1-\cos\theta) \tag{31}$$

[Sim from https://sciencesims.com/sims/cycloid-wheel]

3. Euler-Lagrange Equation

$$\frac{\partial F}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right) = 0 \tag{32}$$

$$I = \int_{x_a}^{x_b} F\left[y(x), y'(x), x\right] dx \tag{33}$$



The right path minimizes the integral. The wrong path does not.

Imagine a *wrong* curve, shown in the dotted red line.

We can call it

$$Y(x) = y(x) + \eta(x) \tag{34}$$

η is just the difference between the *right* path ($y(x)$) and the *wrong* path.

Since they both start and end at the same two points:

$$\eta(x_a) = \eta(x_b) = 0 \tag{35}$$

Introduce a parameter α:

$$Y(x) = y(x) + \alpha\eta(x) \tag{36}$$

If we set $\alpha = 0$ in (36), then we can see that $Y(x)$ is the *right* path. Thus $I(\alpha)$ is a minimum when $\alpha = 0$.

Now show that

$$\frac{dI}{d\alpha} = 0 \text{ for } \alpha = 0 \tag{37}$$

The integral $I(\alpha)$ becomes, when written out:

$$I(\alpha) = \int_{x_a}^{x_b} F(Y, Y', x)dx \tag{38}$$

or

$$I(\alpha) = \int_{x_a}^{x_b} F(y + \alpha\eta,\ y' + \alpha\eta',\ x)\ dx \tag{39}$$

Now we evaluate $\frac{\partial F}{\partial \alpha}$

$$\frac{\partial F(y + \alpha\eta,\ y' + \alpha\eta',\ x)}{\partial \alpha} \tag{40}$$

The multivariate chain rule:

$$\frac{\partial F}{\partial \alpha} = \frac{\partial F}{\partial Y}\frac{dY}{d\alpha} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\alpha} + \frac{\partial F}{\partial x}\frac{dx}{d\alpha} \tag{41}$$

Since x doesn't depend on α:

$$\frac{dx}{d\alpha} = 0$$

and we will have two terms:

$$\frac{\partial F}{\partial \alpha} = \eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'} \tag{42}$$

Thus, $dI/d\alpha$:

$$\frac{\partial I}{\partial \alpha}=\int_{x_a}^{x_b}\frac{\partial F}{\partial \alpha}dx=\int_{x_a}^{x_b}\left(\eta\frac{\partial F}{\partial y}+\eta'\frac{\partial F}{\partial y'}\right)dx=0 \tag{43}$$

This condition is true for any $\eta(x)$ that shares the same start/end points, as in (37).

Integration by Parts

$$\int_{x_a}^{x_b}\eta'\frac{\partial F}{\partial y'}dx=\left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x_a}^{x_b}-\int_{x_a}^{x_b}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)dx \tag{44}$$

Because of (35), the first term on the R.H.S. of (44) is 0, and we have:

$$\int_{x_a}^{x_b}\eta(x)\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}\right)dx=0 \tag{45}$$

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0 \tag{46}$$

Fundamental Lemma of Calculus of Variations:
if

$$\int \eta(x)g(x)dx=0 \tag{47}$$

then it can be shown that $g(x)$ must be zero.

if not, then assume $g(x)$ is non-zero between x_a and x_b.

If I is an extremum:

$$I=\int_{x_a}^{x_b}F\left[y(x),y'(x),x\right]dx \tag{48}$$

then

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0 \tag{49}$$

How will we use this?

$$I=\int\left(\frac{1}{2}mv^2-U\right)dt=\int(T-U)\,dt \tag{50}$$

where

$$T=\frac{1}{2}mv^2 \tag{51}$$

and (for gravity for example)

$$U=mgy \tag{52}$$