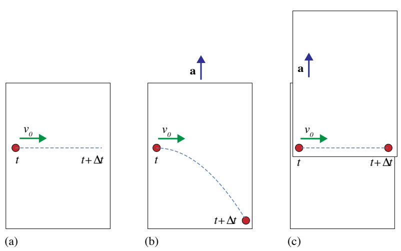
Accelerating Frames

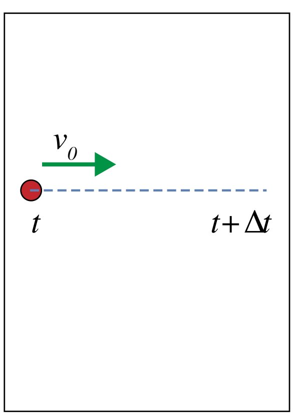
What do we do if we happen to be in a non-inertial frame?

1. Linearly Accelerating Frame



a) A ball thrown sideways with an initial velocity v_0 in a non-accelerating ship/frame. b) the same trajectory from inside the accelerating ship. c) the same trajectory from out side the accelerating ship.

Inertial Observer



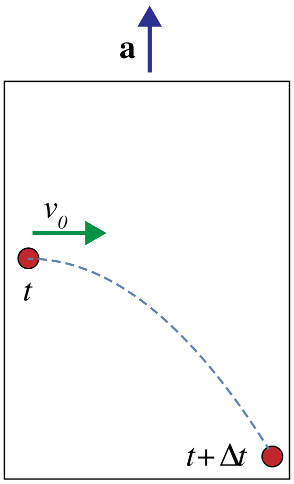
The external observer sees no acceleration in the motion of the ball

The external observer will see the ball move in a straight line.

$$L = T - U = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) \tag{1}$$

which when processed will lead to:

$$\ddot{y} = 0 \tag{2}$$



The observer *in* the ship sees something very different.

Let $\mathbf{r}$ be the position in the inertial (external) frame, and $\mathbf{r}'$ in the ship (non-inertial) frame. These can be related through the acceleration of the ship w.r.t the external frame: a_s:

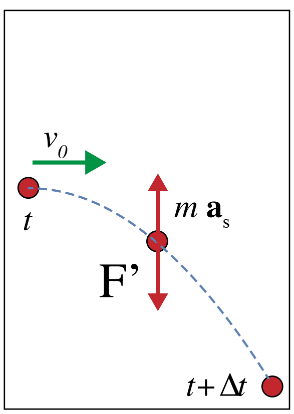
$$\mathbf{r} = \mathbf{r}' + \frac{1}{2}\mathbf{a}_s t^2 \tag{3}$$

Velocities are likewise found to be (through differentiation w.r.t t):

$$\mathbf{v} = \mathbf{v}' + \mathbf{a}_s t \tag{4}$$

and acceleration:

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}_s \tag{5}$$



The non-inertial observer sees a new force: $\mathbf{F}'$

In the inertial frame: $\mathbf{F} = m\mathbf{a}$, or, considering the two frames:

$$\mathbf{F} = m\mathbf{a} = m\left(\mathbf{a}' + \mathbf{a}_s\right) \tag{6}$$

or, considering a new $\mathbf{F}'$:

$$\mathbf{F}' = m\mathbf{a}' = \mathbf{F} - m\mathbf{a}_s \tag{7}$$

So, for observers inside the non-inertial frame:

$$\mathbf{F}' = \mathbf{F} + \mathbf{F}_{\text{pseudo}} \tag{8}$$

This observed force, $\mathbf{F}'$ is the sum of any *real* forces and the *Pseudo-Forces* created by the accelerating reference frame.

2. Meow

Example: **effective gravity** for the accelerating ship:

$$\mathbf{F} = 0 \tag{9}$$

but

$$\mathbf{F}' \neq 0 = 0 - m\mathbf{a}_s = m\mathbf{a}' \tag{10}$$

thus:

$$\mathbf{a}' = -\mathbf{a}_s \tag{11}$$

In this simple case, we can call our pseudo-force, the **effective gravity**:

$$\mathbf{g}_{\text{eff}} = -\mathbf{a}_s \tag{12}$$

Everything can be done that same as before if we include this in our set-up.

What about the Lagrangian approach?

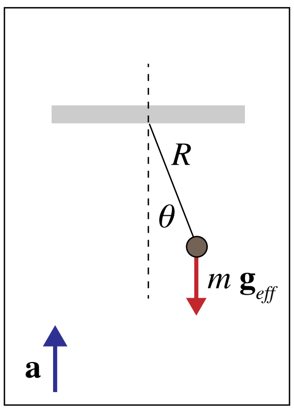
$$L = T' - U' \tag{13}$$

where T' is the kinetic energy in the non-inertial frame, and U' would include and pseudo-potentials:

$$L = \frac{1}{2}m\left(\dot{x}'^2 + \dot{y}'^2\right) - mg_{\text{eff}}y' \tag{14}$$

Example Problem #1:

2.1 The pendulum in an accelerating frame



A pendulum in an accelerating spacecraft

Find the Lagrangian for the non-inertial observer:

$$L = T' - U' = \frac{1}{2}mR^2\dot{\theta}'^2 - mg_{\text{eff}}R\left(1 - \cos\theta'\right) \tag{15}$$

Then crank out the E-L equation:

$$\frac{\partial L}{\partial \theta'} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}'}\right) = 0 \tag{16}$$

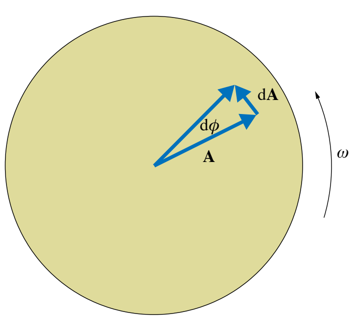
This leads to:

$$-mg_{\text{eff}}R\sin\theta' - \frac{d}{dt}mR^2\dot{\theta}' = 0 \tag{17}$$

or,

$$\ddot{\theta}' + \frac{g_{\text{eff}}}{R}\sin\theta' = 0 \tag{18}$$

3. Rotating Frames



A vector in a rotating frame

Assume vector $\mathbf{A}$ is at rest in the rotating frame. As the frame rotates, the vector $\mathbf{A}$ will also rotate through an angle $d\phi$. The change in $\mathbf{A}$ can then be expressed by the cross-product:

$$d\mathbf{A} = d\boldsymbol{\phi} \times \mathbf{A} \tag{19}$$

($d\boldsymbol{\phi}$ points *out of the page*)

Now, what if $\mathbf{A}$ is *also* changing in the rotating frame?

$$d\mathbf{A}_{\text{in}} = d\mathbf{A}_{\text{rot}} + d\boldsymbol{\phi} \times \mathbf{A} \tag{20}$$

Next, consider the time derivatives:

$$\left.\frac{d\mathbf{A}}{dt}\right|_{\text{in}} = \left.\frac{d\mathbf{A}}{dt}\right|_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{A} \tag{21}$$

where $\boldsymbol{\omega} = \frac{d\boldsymbol{\phi}}{dt}$

Helpful to consider this an operator:

$$\left.\frac{d}{dt}\right|_{\text{in}} = \left.\frac{d}{dt}\right|_{\text{rot}} + \boldsymbol{\omega}\times \tag{22}$$

This can *operate* on any vector and will transform from one coordinate system to another: inertial -> rotating.

Now we express the velocity in the inertial frame in terms of the velocity in the rotating frame (let our position be $\mathbf{r}$):

$$\mathbf{v}_{\text{in}} = \mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r} \tag{23}$$

Using this, we can construct the Lagrangian:

$$L = \frac{1}{2}m\mathbf{v}^2 - U(\mathbf{r}) \tag{24}$$

$$L = \frac{1}{2}m\mathbf{v}_{\text{in}}^2 - U(r) = \frac{1}{2}m(\mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r})^2 - U(\mathbf{r}) \tag{25}$$

Now for some vector math:

$$\begin{aligned}\frac{1}{2}m(\mathbf{v}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r})^2 &= \frac{1}{2}m\mathbf{v}_{\text{rot}}^2 + m\mathbf{v}_{\text{rot}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2}m(\boldsymbol{\omega} \times \mathbf{r})^2 \\ &= \frac{1}{2}m\mathbf{v}_{\text{rot}}^2 + m\mathbf{v}_{\text{rot}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2\end{aligned}$$

The Binet–Cauchy identity:

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}) \tag{26}$$

Thus, in the rotating frame, were we to construct a Lagrangian based on our measurements, we would obtain:

$$L = \frac{1}{2}m\mathbf{v}_{\text{rot}}^2 + m\mathbf{v}_{\text{rot}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2 - U(\mathbf{r}) \tag{27}$$

This is clearly not just $\frac{1}{2}m\mathbf{v}_{\text{rot}}^2$

Find the equations of motion using the Lagrangian:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_{\text{rot}}^i}\right) = \frac{\partial L}{\partial r_{\text{rot}}^i} \tag{28}$$

Here, i implies x, y, and z in the rotating frame.

$$L = \frac{1}{2}m\mathbf{v}_{\text{rot}}^2 + m\mathbf{v}_{\text{rot}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2 - U(\mathbf{r}) \tag{29}$$

So, taking the time derivative of $\frac{\partial L}{\partial \dot{r}_{\text{rot}}^i}$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}^i_{\text{rot}}}\right) = m\ddot{r}^i_{\text{rot}} + m(\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}})^i + m\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right)^i \tag{30}$$

$$L = \frac{1}{2}m\mathbf{v}^2_{\text{rot}} + m\mathbf{v}_{\text{rot}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) + \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2 - U(\mathbf{r}) \tag{31}$$

For the R.H.S. ($\frac{\partial L}{\partial r^i_{\text{rot}}}$): first rewrite one term as:

$$m\mathbf{v}_{\text{rot}} \cdot (\boldsymbol{\omega} \times \mathbf{r}) = m\mathbf{r} \cdot (\mathbf{v}_{\text{rot}} \times \boldsymbol{\omega}) \tag{32}$$

Now the Lagrangian is:

$$L = \frac{1}{2}m\mathbf{v}^2_{\text{rot}} + m\mathbf{r} \cdot (\mathbf{v}_{\text{rot}} \times \boldsymbol{\omega}) + \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2 - U(\mathbf{r}) \tag{33}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \tag{34}$$

$$L = \frac{1}{2}m\mathbf{v}^2_{\text{rot}} + m\mathbf{r} \cdot (\mathbf{v}_{\text{rot}} \times \boldsymbol{\omega}) + \frac{1}{2}m\omega^2 r^2 - \frac{1}{2}m(\boldsymbol{\omega} \cdot \mathbf{r})^2 - U(\mathbf{r}) \tag{35}$$

Leads to:

$$\frac{\partial L}{\partial r^i_{\text{rot}}} = m(\mathbf{v}_{\text{rot}} \times \boldsymbol{\omega})^i_{\text{rot}} + m\omega^2 r^i_{\text{rot}} - m\,(\boldsymbol{\omega} \cdot \mathbf{r})\,\omega^i - \frac{\partial U(\mathbf{r})}{\partial r^i_{\text{rot}}} \tag{36}$$

Thus our equation of motion in the rotating system is:

$$m\mathbf{a} = m\omega^2 \mathbf{r}_{\text{rot}} - m\,(\boldsymbol{\omega} \cdot \mathbf{r})\,\boldsymbol{\omega}_{\text{rot}} - 2m(\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}})_{\text{rot}} - m\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right)_{\text{rot}} - \nabla_{\text{rot}} U(\mathbf{r}) \tag{37}$$

Combine the first two terms:

$$m\omega^2 \mathbf{r}_{\text{rot}} - m\,(\boldsymbol{\omega} \cdot \mathbf{r})\,\boldsymbol{\omega}_{\text{rot}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})_{\text{rot}} \tag{38}$$

(using a vector identity.)

Vector identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{39}$$

Finaaaaly,

$$\mathbf{F}_{\text{rot}} = \mathbf{F}_{\text{in}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})_{\text{rot}} - 2m(\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}})_{\text{rot}} - m(\dot{\boldsymbol{\omega}} \times \mathbf{r})_{\text{rot}} \tag{40}$$

where $\mathbf{F}_{\text{in}} = -\nabla_{\text{rot}} U(\mathbf{r})$ is the sum of *real* forces acting in the *inertial frame*

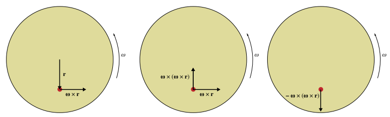
What are these pseudo-forces?,

$$\mathbf{F}_{\text{rot}} = \mathbf{F}_{\text{in}} \underbrace{-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})_{\text{rot}}}_{\text{centrifugal}} \underbrace{-2m(\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}})_{\text{rot}}}_{\text{Coriolis}} \underbrace{-m(\dot{\boldsymbol{\omega}} \times \mathbf{r})_{\text{rot}}}_{\text{Euler}} \tag{41}$$

4. Pseudo-Forces

4.2 Centrifugal

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})_{\text{rot}} \tag{42}$$



Verify the direction of the centrifugal pseudo-force

4.3 Coriolis

$$-2m\left(\boldsymbol{\omega}\times\mathbf{v}_{\text{rot}}\right) \tag{43}$$

The Coriolis force acts when the object is moving in a direction that is not parallel to $\boldsymbol{\omega}$

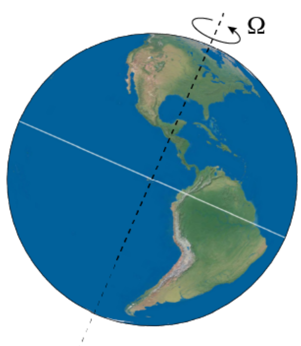
4.4 Euler

$$-m(\dot{\boldsymbol{\omega}}\times\mathbf{r})_{\text{rot}} \tag{44}$$

This pseudo-force only arises when $\boldsymbol{\omega}$ is changing, i.e. the speed of rotation is changing.

[Sim from https://sciencesims.com/sims/rotating-ref-frame/]

5. Earth as rotating reference frame



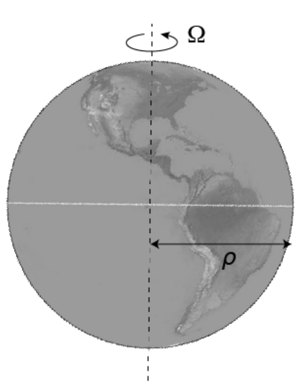
The Earth Rotating on its axis

Earth takes 24 hours to rotate once.

(Actually, a little bit less: sidereal day is 23h56'04'')

$$\begin{aligned}\Omega &= \frac{2\pi}{24\ \text{h}\times 3600\ \text{s/h}}\left(\frac{366.5}{365.5}\right)\\ &= 7.292\times 10^{-5}\ \text{s}^{-1}\end{aligned}$$

[Sim from https://sciencesims.com/sims/solar-vs-sidereal/]



Let ρ be the distance from the rotation axis.

The **Centrifugal pseudo-force** will be directed outward and proportional to the distance from the rotation axis ρ.

$$\mathbf{F}_{\text{cent}} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})_{\text{rot}} \tag{45}$$

can be simplified to:

$$F = m\Omega^2\rho \tag{46}$$

Taking the radius at the equator to be 6378 km:

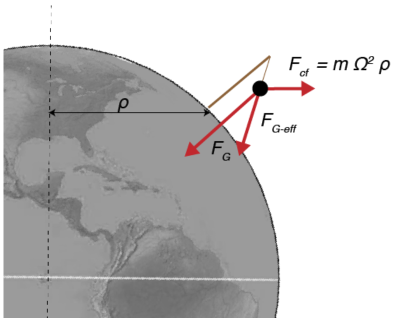
$$6.374\times 10^{6}\ \text{m}\times\left(7.292\times 10^{-5}\ \text{s}^{-1}\right)^2 = 0.033914\ \text{m/s}^2 \tag{47}$$

Which is about 0.3% of the normal g.

Thus, the *effective* gravity would be:

$$g_{\text{eff}} = g - \Omega^2 \rho \tag{48}$$

The value of ρ depends on latitude, and you can easily see how at the poles, this contribution should be zero.

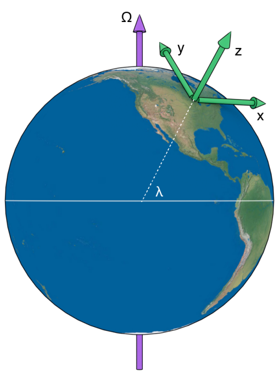


Forces on a plumb bob in the northern hemisphere

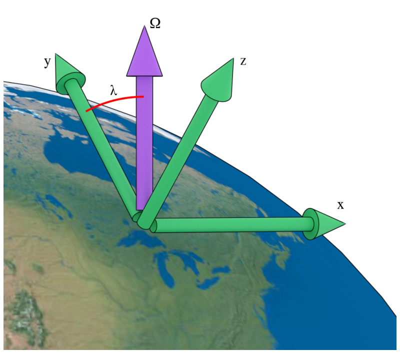
Since these are all vectors, we can see the effects on a plumb-bob hanging in the north hemisphere:

$$\mathbf{F}_{\text{G-eff}} = \mathbf{F}_G - \mathbf{F}_{\text{cf}} \tag{49}$$

6. Coriolis



A cartesian set on the surface of the Earth.



A cartesian set on the surface of the Earth.

Express $\boldsymbol{\Omega}$ in terms of cartesian vectors:

$$\boldsymbol{\Omega} = \Omega\left(\hat{\mathbf{y}}\cos\lambda + \hat{\mathbf{z}}\sin\lambda\right) \tag{50}$$

A particle will have velocity components:

$$\mathbf{v} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}} \tag{51}$$

Thus we can compute the Coriolis pseudo-force vector:

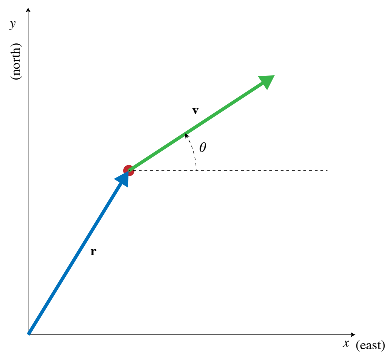
$$\begin{aligned} \mathbf{F}_{\text{Cor}} &= -2m\boldsymbol{\Omega} \times \mathbf{v} && (52) \\ &= 2m\Omega\left[(\dot{y}\sin\lambda - \dot{z}\cos\lambda)\,\hat{\mathbf{x}} - \dot{x}\sin\lambda\,\hat{\mathbf{y}} + \dot{x}\cos\lambda\,\hat{\mathbf{z}}\right] && (53) \end{aligned}$$

If there is no vertical motion, i.e. $\dot{z} = 0$:

$$\mathbf{F}_{\text{Cor, horiz}} = 2m\Omega \sin\lambda \left(\dot{y}\hat{\mathbf{x}} - \dot{x}\hat{\mathbf{y}}\right) \tag{54}$$

but $\mathbf{v} = (\dot{x}, \dot{y}) = (v\cos\theta, v\sin\theta)$ so:

$$\mathbf{F}_{\text{Cor, horiz}} = 2m\Omega \sin\lambda v \left(\sin\theta\hat{\mathbf{x}} - \cos\theta\hat{\mathbf{y}}\right) \tag{55}$$

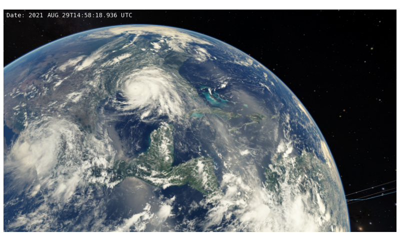


The $\mathbf{r}$ and $\mathbf{v}$ and $\boldsymbol{\theta}$ vectors

or, in terms of the vector: $\hat{\boldsymbol{\theta}}$

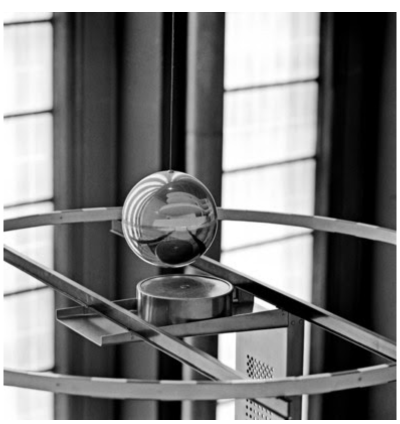
$$\mathbf{F}_{\text{Cor, horiz}} = -2m\Omega \sin\lambda v\hat{\boldsymbol{\theta}} \tag{56}$$

1. The Coriolis force therefore pushes moving objects the right of their velocity vector if $\sin\lambda > 0$ (i.e. northern hemisphere)
2. The Coriolis force therefore pushes moving objects the left of their velocity vector if $\sin\lambda < 0$ (i.e. southern hemisphere)



Hurricane Ida

6.5 Foucault Pendulum



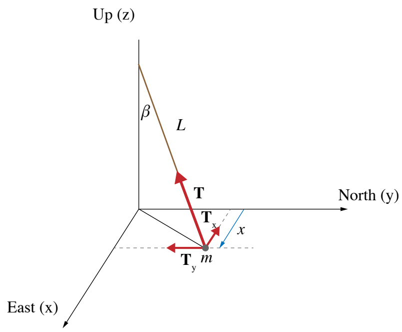
The forces in the rotating frame of the Earth:

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g}_0 + m\left(\boldsymbol{\Omega}\times\mathbf{r}\right)\times\boldsymbol{\Omega} + 2m\dot{\mathbf{r}}\times\boldsymbol{\Omega} \tag{57}$$

Combine $\mathbf{g}_0$ and $m\left(\boldsymbol{\Omega}\times\mathbf{r}\right)\times\boldsymbol{\Omega}$ to express in terms of the observed gravitational force:

$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g} + 2m\dot{\mathbf{r}}\times\boldsymbol{\Omega} \tag{58}$$

As usual, let's restrict ourselves to *small oscillations*



The Pendulum

This let's us say that

$$T \approx mg \tag{59}$$

since $T_z = T \cos\beta \approx T$

Next, we need to examine the x and y components

Looking at the figure:

$$\frac{T_x}{T} = -\frac{x}{L} \text{ and } \frac{T_y}{T} = -\frac{y}{L} \tag{60}$$

therefore:

$$T_x = \frac{-mgx}{L} \text{ and } T_y = \frac{-mgy}{L} \tag{61}$$

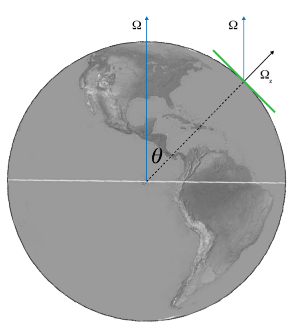
$$m\ddot{\mathbf{r}} = \mathbf{T} + m\mathbf{g} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} \tag{62}$$

Then we have two equations of motion, after some algebra

$$\ddot{x} = \frac{-gx}{L} + 2\dot{y}\Omega\cos\theta \tag{63}$$

$$\ddot{y} = \frac{-gy}{L} - 2\dot{x}\Omega\cos\theta \tag{64}$$

θ is the colatitude of the experiment.



The Ω_z and colatitude angle θ.

A little more cleaning up and we have:

$$\ddot{x} - 2\dot{y}\Omega_z + \omega_0^2 x = 0 \tag{65}$$

$$\ddot{y} + 2\dot{x}\Omega_z + \omega_0^2 y = 0 \tag{66}$$

ω is the natural frequency of the pendulum: $\omega = \sqrt{g/L}$ and $\Omega\cos\theta$ is the z-component of the angular rotation of the Earth.

Let's solve these.

$$\ddot{x} - 2\dot{y}\Omega_z + \omega_0^2 x = 0 \tag{67}$$

$$\ddot{y} + 2\dot{x}\Omega_z + \omega_0^2 y = 0 \tag{68}$$

Start by defining a complex number:

$$\eta = x + iy \tag{69}$$

And multiply the $\ddot{y}$ equation by i, then add it to the $\ddot{x}$ equation:

$$\ddot{\eta} + 2i\Omega_z\dot{\eta} + \omega_0^2\eta = 0 \tag{70}$$

Now we have a *second-order, linear, homogeneous* differential equation.

This implies two indepedent solutions

Try: $\eta(t) = e^{-i\alpha t}$

Leads to:

$$\alpha^2 - 2\Omega_z \alpha - \omega_0^2 = 0 \tag{71}$$

or

$$\alpha = \Omega_z \pm \sqrt{\Omega_z^2 + \omega_0^2} \tag{72}$$

Thus:

$$\eta = e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right) \tag{73}$$

With $x = A$ and $y = 0$ at $t = 0$, the pendulum is released from rest ($v_{x0} = v_{y0} = 0$) and we can obtain:

$$C_1 = C_2 = \frac{A}{2} \tag{74}$$

$$\eta = e^{-i\Omega_z t}\left(C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t}\right)$$

at $t = 0$, we can say from the initial conditions: $\eta(0) = A$ and $\dot{\eta}(0) = 0$

Thus, $C_1 + C_2 = A$ and by taking the first time derivative:

$$-i\Omega_z\left(C_1 + C_2\right) + i\omega_0\left(C_1 - C_2\right) = 0$$

Solving these two equation for C_1

$$C_1 = \frac{A}{2}\left(1 + \frac{\Omega_z}{\omega_0}\right) \approx \frac{A}{2} \tag{75}$$

and

$$C_2 = \frac{A}{2}\left(1 - \frac{\Omega_z}{\omega_0}\right) \approx \frac{A}{2} \tag{76}$$

Since $\Omega_z << \omega_0$ we can safely make this approximation.

$$\begin{aligned} \eta(t) &= x(t) + iy(t) & (77) \\ &= Ae^{-i\Omega_z t}\cos\omega_0 t & (78) \end{aligned}$$

Foucault Pendulum - Time Lapse

https://www.youtube.com/watch?v=xmqjokCwNQs