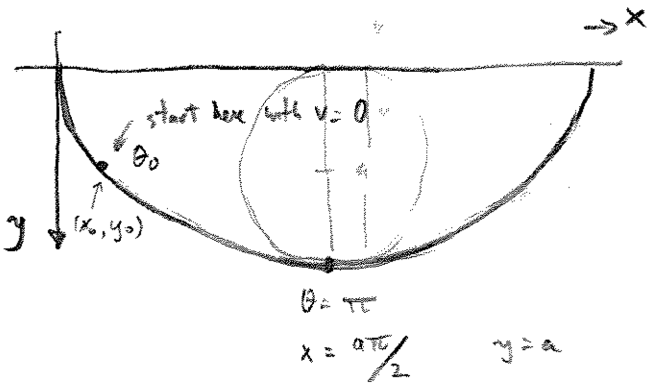
1.

$x = \frac{a}{2}(\theta - \sin\theta)$

$y = \frac{a}{2}(1 - \cos\theta)$



find E of partical

$E = \frac{1}{2}mv^2 + mg(-y) = -mgy_0$

$\therefore \quad v = (2g(y - y_0))^{1/2}$

$t = \int \frac{ds}{v} \quad \leftarrow \quad \left(v = \frac{ds}{dt} \right)$

$t = \int_{\theta=\theta_0}^{\pi} \frac{(dx^2 + dy^2)^{1/2}}{(2g(y - y_0))^{1/2}}$, now find $\frac{dx}{d\theta}$ of $x = \frac{a}{2}(\theta - \sin\theta)$

$\frac{dy}{d\theta}$ of $y = \frac{a}{2}(1 - \cos\theta)$

$dx = \frac{a}{2}(1 - \cos\theta)\, d\theta \;\; ; \quad dy = \frac{a}{2}(\sin\theta)\, d\theta$

which leads to $\quad dx^2 + dy^2 = \frac{a^2}{2}(1 - \cos\theta)\, d\theta^2$

and $\quad y - y_0 = \frac{a}{2}(\cos\theta - \cos\theta_0)$

a) $\quad t = \int_{\theta=\theta_0}^{\pi} \frac{\sqrt{\frac{a^2}{2}(1 - \cos\theta)\, d\theta^2}}{\sqrt{2g\frac{a}{2}(\cos\theta - \cos\theta_0)}} = \sqrt{\frac{a}{2g}} \int \sqrt{\frac{1 - \cos\theta}{\cos\theta - \cos\theta_0}}\, d\theta$

b) now, $\quad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} = 2\sin^2\varphi \quad$ if $\quad \varphi = \theta/2 \quad$ and $\quad \frac{d\theta}{d\varphi} = 2 \quad \therefore d\theta = 2\, d\varphi$

$t = \sqrt{\frac{a}{2g}} \int_{\varphi_0}^{\pi/2} \frac{2\sin\varphi}{\sqrt{\cos^2\varphi_0 - \cos^2\varphi}}\, d\varphi \qquad$ now, if $\cos\varphi = z \quad \therefore \frac{dz}{d\varphi} = -\sin\varphi$

sub let $\frac{z}{z_0} = v$ $\therefore dz = z_0\, dv$

$t = 2\sqrt{\frac{a}{2g}} \int_0^{z_0} \frac{dz}{(z_0^2 - z^2)^{1/2}} = 2\sqrt{\frac{a}{2g}} \int_0^1 \frac{dv}{(1 - v^2)} = \pi\sqrt{\frac{a}{2g}}$

2

Surface defined by $z = x^{3/2}$

for 3d, arc length would be

$$S = \int \sqrt{dx^2 + dy^2 + dz^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx \quad , \text{ let } y' = \frac{dy}{dx}$$

and since $\frac{dz}{dx} = \frac{3}{2} x^{1/2}$, $\left(\frac{dz}{dx}\right)^2 = \frac{9}{4} x$, so

$$S = \int \underbrace{\sqrt{1 + y'^2 + \frac{9}{4}x}}_{\text{call this our } F \text{ and use Euler-Lagrange}}\, dx$$

$$\underbrace{\frac{\partial F}{\partial y}}_{\substack{\text{this is} \\ 0}} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

so this is a constant

$$\underbrace{\frac{\partial F}{\partial y'} = \frac{1}{2}\left(1 + y'^2 + \frac{9}{4}x\right)^{-1/2} 2y' = C}_{\text{solve for } y' \ldots}$$

Remember, Constant

$$y' = \boxed{\sqrt{\frac{C^2}{1-C^2}}} \sqrt{1 + \frac{9}{4}x} = \frac{dy}{dx}$$

$$\therefore \int dy = \sqrt{\frac{C^2}{1-C^2}} \int \sqrt{1 + \frac{9}{4}x}\, dx$$

$$y = \underbrace{\frac{c}{\sqrt{1-c^2}} \frac{8}{27}}_{\text{put all in } A} \left(1 + \frac{9}{4}x\right)^{3/2} + B$$

need another constant after integrating for dx

$$\therefore \quad y = A\left(1 + \frac{9}{4}x\right)^{3/2} + B$$

now, since our two points are $(0,0)$ & $(1,1)$

$$0 = A(1)^{3/2} + B \qquad \text{and then} \qquad 1 = A\left(1 + \frac{9}{4}(1)\right)^{3/2} - A$$

or $A = -B$

solve for A

$$A = \frac{1}{\left(1 + \frac{9}{4}\right)^{3/2} - 1} = \frac{8}{13^{3/2} - 8}$$

thus our functio $y(x)$ will be

$$\boxed{y(x) = \frac{8}{13^{3/2} - 8}\left(\left(1 + \frac{9}{4}x\right)^{3/2} - 1\right)}$$

and $z(x)$ we know

$$z(x) = x^{3/2}$$

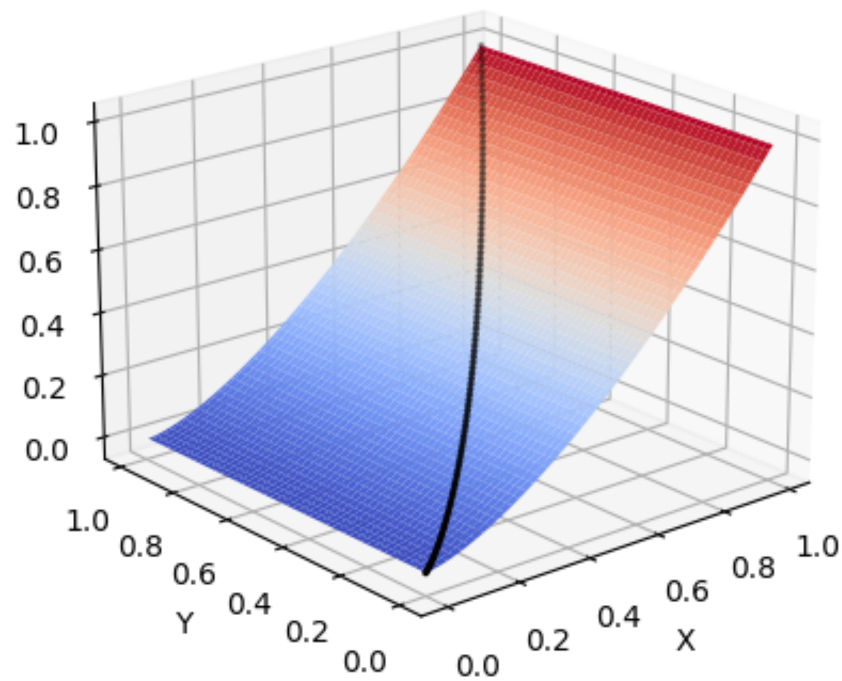
this is the curve on the surface that is the shortest path from $(0,0,0)$ to $(1,1,1)$

```
import matplotlib.pyplot as plt
import numpy as np

from matplotlib import cm
```

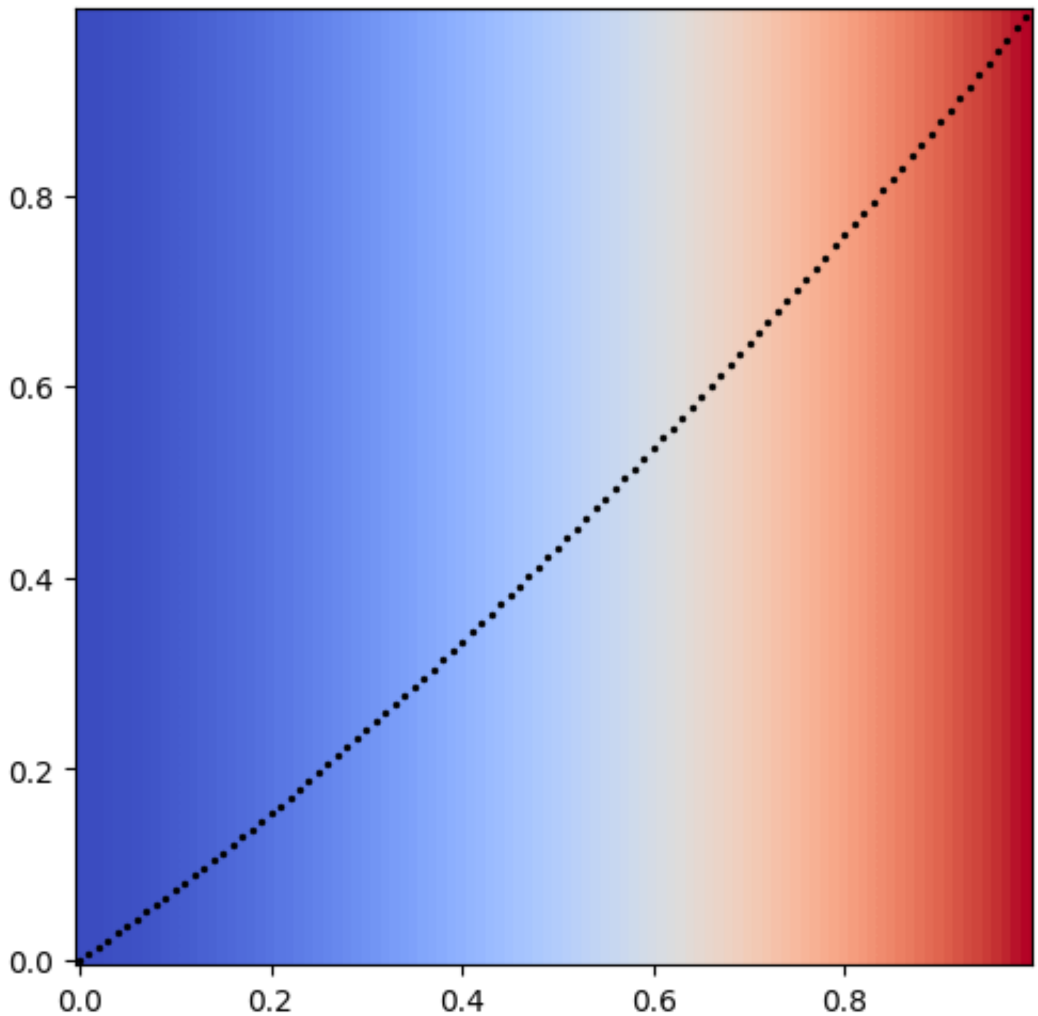
```
# Make data
dx = 0.01
dy = 0.01
X = np.arange(0, 1, dx)
Y = np.arange(0, 1, dy)
X, Y = np.meshgrid(X, Y,indexing = 'ij')

Z = np.power(X,1.5)
```

```
xLine = np.arange(0,1,.01)
yLine = 8/(np.power(13,1.5)-8)*(np.power((1+(9/4)*xLine),1.5)-1)
zLine = np.power(xLine,1.5)
```

```
# Plot the surface
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
ax.plot_surface(X, Y, Z, vmin=Z.min(), cmap=cm.coolwarm)
ax.scatter(xLine, yLine, zLine, marker='o',s=2,color='black')
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_zlabel('Z')
ax.view_init(elev=20., azim=230, roll=0)
plt.show()
```

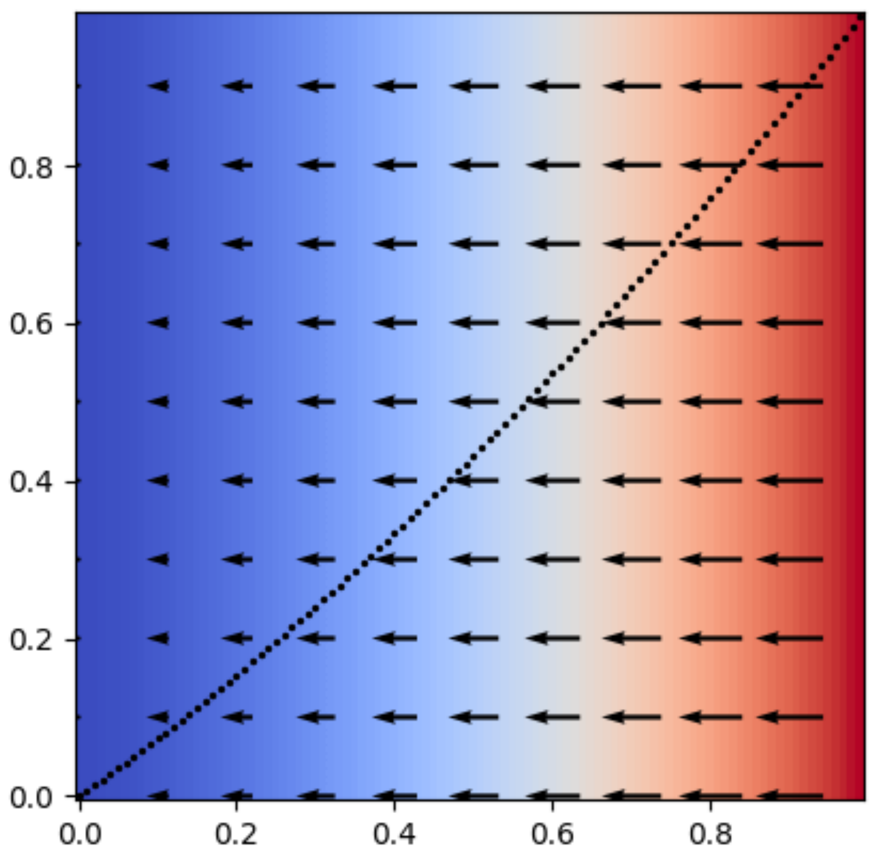


```
fig1, ax = plt.subplots(layout='constrained')
ax.pcolor(X, Y, Z, cmap=cm.coolwarm)
ax.scatter(xLine, yLine, marker='o',s=2,color='black')
ax.set_aspect('equal')
plt.show()
```




```
gradx, grady = np.gradient(Z,dx,dy,edge_order=2)
```

```
skip=10
fig, ax = plt.subplots()
ax.pcolor(X, Y, Z, cmap=cm.coolwarm)
q = ax.quiver(X[0::skip,0::skip], Y[0::skip,0::skip], -gradx[0::skip,0::skip], -gra


ax.scatter(xLine, yLine, marker='o',s=2,color='black')

ax.set_aspect('equal')

plt.show()
```



3. Simple Projectile

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \& \quad U = mgz$$

$$\mathcal{L} = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

do: Euler Lagrange x 3

$$\underbrace{\frac{\partial L}{\partial x}}_{=0} - \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)}_{m\ddot{x}} = 0$$

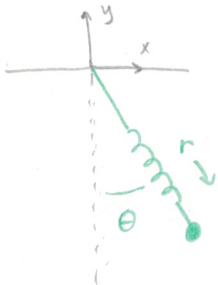
$$\underbrace{\frac{\partial L}{\partial y}}_{=0} - \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)}_{m\ddot{y}} = 0$$

$$\underbrace{\frac{\partial L}{\partial z}}_{\neq 0} - \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)}_{m\ddot{z}} = 0$$

thus

$$m\ddot{x} = 0, \quad m\ddot{y} = 0 \quad \& \quad m\ddot{z} = -mg$$

4. Spring / Pendulum



a) $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$ [cartesian]

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$ [polar)

kinetic

$U = \underbrace{\frac{1}{2} k (r-\ell)^2}_{\text{spring, note when } r=\ell \text{, } U_{sp}=0} - \underbrace{mgr\cos\theta}_{\text{gravitational potential}}$

b) Lagrangian is thus:

$$\mathcal{L} = T - U = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} k (r-\ell)^2 + mgr\cos\theta$$

use Euler-Lagrange

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \quad \& \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$\frac{\partial L}{\partial r} = \frac{1}{2} m 2 r \dot{\theta}^2 - \frac{1}{2} k 2 (r-\ell)(1) + mg\cos\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

$\therefore \boxed{m\ddot{r} = mr\dot{\theta}^2 - k(r-\ell) + mg\cos\theta}$

$\frac{\partial L}{\partial \theta} = -gmr\sin\theta$

$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m 2 \dot{\theta} r^2$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m(\ddot{\theta} r^2 + \dot{\theta} 2 r \dot{r})$

$\therefore \boxed{r\ddot{\theta} = -2\dot{r}\dot{\theta} - g\sin\theta}$

c) if $\dot{r}, \ddot{r}, \dot{\theta}, \ddot{\theta} = 0$, then

$\theta_0 = 0 \quad \& \quad +k(r-\ell) = mg\cos\theta_0$

$\downarrow$

Equilibrium Length $\left\{ r_0 = \ell + \frac{mg}{k} \right\}$

d) if $\theta, \dot{\theta}$, are small $(\dot{\theta}^2 \approx 0)$ $(\dot{r}\dot{\theta} \approx 0)$

$\therefore \ddot{r} = -\frac{k}{m}(r-\ell) + g$ [let $\rho = r - r_0$]

displacement from equibrium.

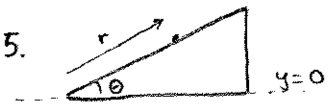
becomes

$\ddot{\rho} = -\frac{k}{m}\rho$ (SHO)

[cos (small) ~ 1]
sin (small) ~ θ

and $\ddot{\theta} = -\frac{g}{r_0}\theta$ (SHO)

5.



first, find the velocity of the mass in polar coordinates:

$$v^2(r,\theta) = \dot{r}^2 + r^2\dot{\theta}^2$$

$$U = mgr\sin\theta$$

thus, our Lagrangian is $L = T - U$

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$$

$$\text{let } \underbrace{\dot{\theta} = \omega}_{\text{constant}} \implies \theta = \omega t$$

thus

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + \omega^2 r^2) - mgr\sin(\omega t)$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \implies \frac{\partial L}{\partial r} = \tfrac{1}{2}m\omega^2 2r - mg\sin(\omega t)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\therefore \quad m\ddot{r} = m\omega^2 r - mg\sin(\omega t)$$

or $\ddot{r} - \omega^2 r = -g\sin(\omega t)$ } Non-homogeneous Diff Eq

General solution $\Rightarrow r = r_H + r_P$ (Homogeneous & Particular)

(Homogeneous) $\longrightarrow \ddot{r} - \omega^2 r = 0 \implies r_H = Ae^{\omega t} + Be^{-\omega t}$

(Particular) $\longrightarrow$ try $r_p = C\sin\omega t$, then $\ddot{r}_p = -C\omega^2\sin(\omega t)$

$\therefore \quad -C\omega^2\sin(\omega t) - \omega^2 C\sin\omega t = -g\sin(\omega t)$ solve for C

$$C = \frac{g}{2\omega^2}, \text{ then } r(t) = Ae^{\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$$

Now, initial conditions $\Rightarrow r(0) = r_0$ & $\dot{r}(0) = 0$

thus $r_0 = A + B$ & $0 = A - B + \frac{g}{2\omega^2} \implies$ Solve for A & B

$$A = \tfrac{1}{2}\left[r_0 - \frac{g}{2\omega^2}\right] \quad \& \quad B = \tfrac{1}{2}\left[r_0 + \frac{g}{2\omega^2}\right]$$

leads to: $r = \frac{1}{2}\left[r_0 - \frac{g}{2\omega^2}\right]e^{\omega t} + \frac{1}{2}\left[r_0 + \frac{g}{2\omega^2}\right]e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$

or : $r(t) = r_0\cosh(\omega t) + \frac{g}{2\omega^2}\left(\sin(\omega t) - \sinh(\omega t)\right)$