CCNY PHYS 35100 - Fall 2024 - Midterm Exam 2 Name Solutions

Nov 20 2024

Instructions: There are 4 problems. Each part of each problem is worth the same amount (except for #1). Please do your work on this test paper. There is extra paper attached to the back if you would like it for rough work, etc. When doing the problems, the more steps you justify, the more likely you are to receive full credit, in other words, please show your work. Show *how you know*, not just *that you know.* No external test aids may be used.

In general, you can assume that the acceleration due to gravity is the regular g and that the systems shown are near the surface of the earth, unless otherwise specified.

Some mathy things that might help

$$a^2 + b^2 = c^2 \neq (a + b)^2$$

1. Some yes/no, multiple choice to warm up. Clearly circle your choice. You don't need to explain your choices here. (1pt each)

i. For a simple conservative system, like a ball falling under the influence of gravity in a vacuum, or a simple spring and mass system, does the value of the Lagrangian change as a function of time? $\boxed{\text{YES}}$ NO

ii. If you drop a rock and it falls to the Earth, does the position of the center of mass of the Earth-Rock pair change due to the motion of the rock? YES $\boxed{\text{NO}}$

iii. For an idealized two-body gravitation system, which of the following quantities can change during an orbit?

a. The total angular momentum.

b. The total energy of the two body system.

$\boxed{\text{c.}}$ The kinetic energy of the smaller object.

d. None of these can change during an orbit.

iv. What was the astronomer Tycho Brahe best known for?

a. Laying the foundations of ellipse geometry

b. Improving Galileo's telescope

c. Discovering the orbit of Neptune

$\boxed{\text{d.}}$ Highly accurate data of stellar positions

v. Kepler's 3rd law says that the semi-major axis of a planet's orbit is proportional to the planet's orbital period, T, raised to what power?:

$\boxed{\text{a.}}$ $T^{2/3}$

b. $T^{3/2}$

c. $T^{1/2}$

d. $T^{1/3}$

vi. Which of the following is the tangential component of a particle's velocity in polar (r, θ) coordinates?

a. $v_\theta = \dot{r}\,\theta$

b. $v_\theta = r^2\,\dot{\theta}$

$\boxed{\text{c.}}$ $v_\theta = r\,\dot{\theta}$

d. $v_\theta = r\,\dot{\theta}^2$

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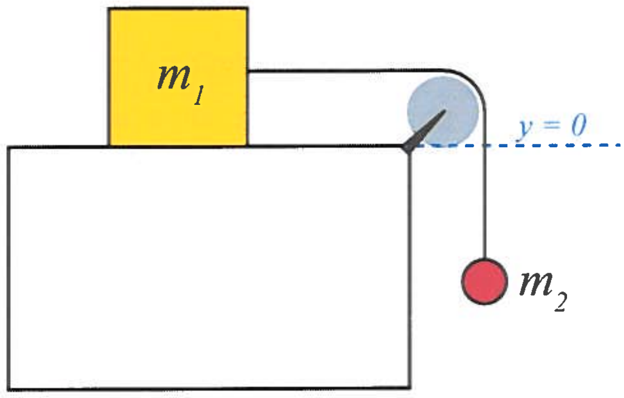
2. Classic Lagrangian



Two objects are connected via a massless string as shown. Block 1 is on a frictionless surface, while mass 2 is hanging below the (also massless) pulley. Using the y coordinate as shown,

a. Write the kinetic energy of the system.

$$T = \frac{1}{2}m_1\dot{y}^2 + \frac{1}{2}m_2\dot{y}^2 = \frac{1}{2}\dot{y}^2(m_1+m_2)$$

b. Write the potential energy of the system.

$$U = -mgy$$

c. Using the Lagrangian methods, find the acceleration of mass 2.

$$L = \frac{1}{2}\dot{y}^2(m_1+m_2) + m_2gy$$

$$\frac{\partial L}{\partial y} = m_2 g$$

$$\frac{\partial L}{\partial \dot{y}} = \dot{y}(m_1+m_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \ddot{y}(m_1+m_2)$$

$$\ddot{y}(m_1+m_2) = m_2 g$$

$$\ddot{y} = \frac{m_2 g}{(m_1+m_2)}$$

d. If mass 2 starts acting like a pendulum while descending down, describe how its frequency of oscillation will be affected, if at all, during the motion.

a) the $\frac{\text{frequency}}{\text{period}}$ will be $\frac{\text{smaller}}{\text{larger}}$ due to the acceleration down

$$\omega = \sqrt{\frac{g_{eff}}{L}} \rightarrow g_{eff} = g\left(1 - \frac{m_2}{m_1+m_2}\right)$$

b) the $\frac{\text{frequency}}{\text{period}}$ will $\frac{\text{decrease}}{\text{increase}}$ due to the increasing length of the string

$$\omega = \sqrt{\frac{g_{eff}}{L}} \rightarrow L \text{ is getting longer.}$$

3. Weird Gravity

A particle of mass m is attracted to a central force that depends inversely on the cube of the distance between the mass and the center, and a multiplicative constant F_0. Our goal is to obtain the effective potential for this system.

a. Write the force acting on the particle in vector format (i.e. $\mathbf{r}$)

$$\vec{F} = -\frac{F_0}{r^3}\hat{r}$$

b. Write the potential energy due to this force.

$$U = -\int \vec{F}\cdot d\vec{r} = -\left(+\frac{F_0 r^{-2}}{2}\right) = -\frac{F_0}{2}\frac{1}{r^2}$$

c. Write the kinetic energy in terms of r and θ

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

d. Solve the Euler-Lagrange equation for the θ coordinate to obtain an expression for angular momentum, call it b.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{F_0}{2}\frac{1}{r^2} \qquad b = mr^2\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\dot{\theta} = \text{constant} = b$$

e. Is angular momentum conserved? Circle one: $\boxed{\text{YES}}$ NO.

f. Solve the Euler-Lagrange equation for r to obtain an expression with $m\ddot{\mathbf{r}}$ on the LHS, and everything else that only depends on position and other things like b, F_0, m and any numerical factors on the RHS.

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - F_0\frac{1}{r^3}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{F_0}{r^3}$$

$$b^2 = m^2r^4\dot{\theta}^2$$

$$m\ddot{r} = mr\left(\frac{b^2}{m^2r^4}\right) - \frac{F_0}{r^3}$$

$$m\ddot{r} = \frac{b^2}{mr^3} - \frac{F_0}{r^3} = \text{Force}$$

g. Knowing that $U = -\int \mathbf{F} \cdot d\mathbf{r}$, determine the effective potential for this system.

$$F = \frac{b^2}{mr^3} - \frac{F_0}{r^3}$$

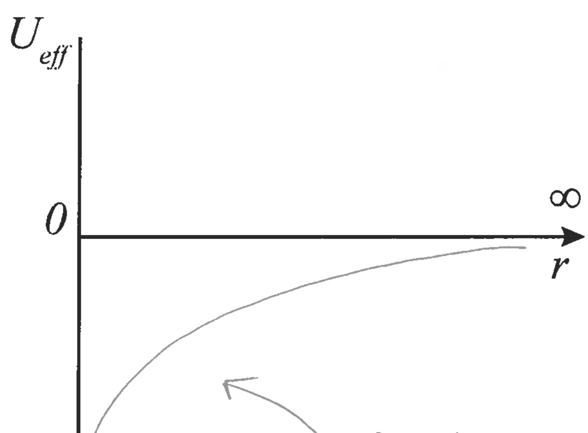
$$-\int F\,dr = -\frac{1}{2}\left(-\frac{b^2}{mr^2} + \frac{F}{r^2}\right) = U_{eff}$$

$$U_{eff} = \frac{1}{2}\left(\frac{b^2}{m} - F_0\right)\frac{1}{r^2}$$

h. Now, depending on the relative values of F_0 and b, you should be able to see that there will be two different curves. Pick one, explain, and sketch it on the axis below.

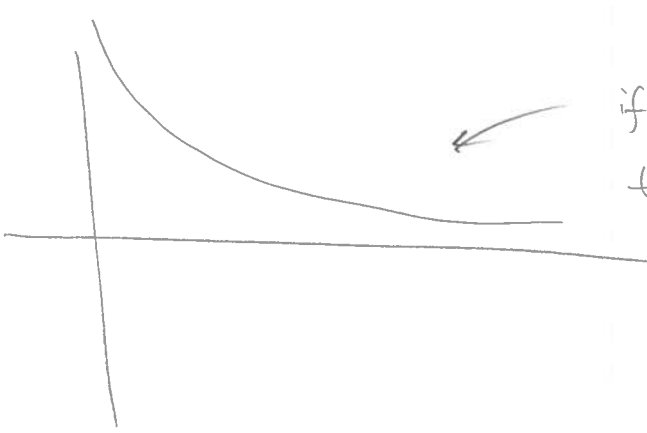
if $\frac{b^2}{m} > F_0$, then $U_{eff} > 0$

if $\frac{b^2}{m} < F_0$, the $U_{eff} < 0$



for the case that $F_0 > \frac{b^2}{m}$.

the mass gets pulled into the center. (spiral)



if $\frac{b^2}{m} > F_0$, then the mass is lost...

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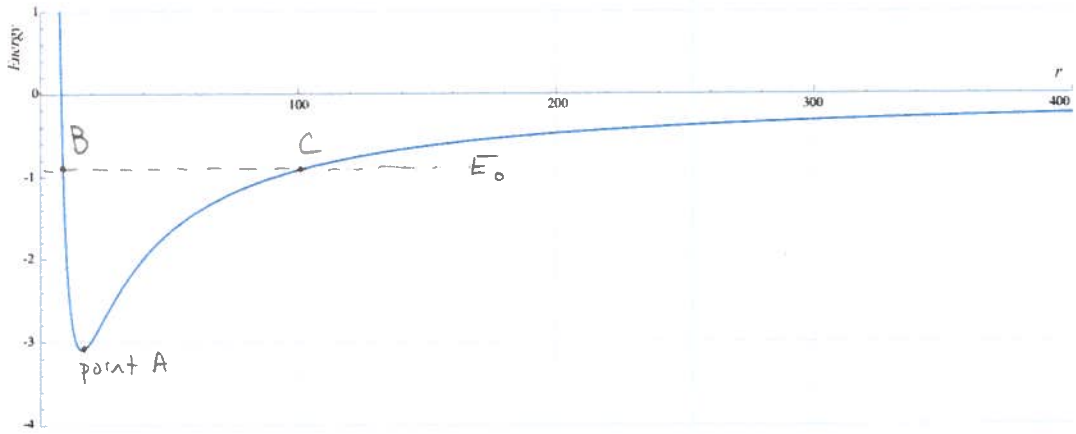
4. Regular Gravity

We have seen this equation for the total energy of a gravitational interaction between masses, m_1 and m_2.

$$\frac{1}{2}\mu\dot{r}^2 + \overbrace{\frac{l^2}{2\mu r^2} - \frac{Gm_1m_2}{r}}^{\to U_{eff}} = E \qquad \frac{dU_{eff}}{dr} = -\frac{l^2}{\mu r^3} + \frac{Gm_1m_2}{r^2} \tag{1.1}$$

We can plot the effective potential part U_{eff} and get a graph like this. We've set $l = 40$, $G = 1.0$, $m_1 = 100$, and $m_2 = 1.0$ for the sake of simplicity.



a. If the orbiter in the system was found to be in an orbit with eccentricity = 0, what would its distance, in terms of (G, l, and the masses) from the central object be?

$$-\frac{l^2}{\mu r^3} + \frac{Gm_1m_2}{r^2} = 0 \quad \text{for circular, } \epsilon = 0, \text{ orbit.}$$

$$\therefore r = \frac{l^2}{\mu G m_1 m_2} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \approx m_2$$

b. Do some simple arithmetic to get an actual number for this r value.

$$r = \frac{40^2}{1 \times 1 \times 100 \times 1} = \frac{16\cancel{00}}{1\cancel{00}} = 16$$

c. Annotate the graph with a *Point A* labeled indicator arrow to show where this is on the graph.

d. Now, if the orbiter was found instead to have an apoapsis* of $r = 100$, what would its total energy be? (Try to get a number with 2 significant figures here)

$$\frac{16\cancel{00}}{2 \times 1 \times 10{,}0\cancel{00}} - \frac{1 \times 100 \times 1}{100} = E$$

$$\frac{16}{200} - 1 = E = \frac{8}{100} - 1 = \frac{8}{100} - \frac{100}{100} = -\frac{92}{100} = -.92$$

e. Annotate the graph with a line labeled E_0 to indicate this total energy value.

f. Annotate the graph with two indicators (*Points B* and *C* respectively) showing the oribiter's closest (periapsis) and furthest (apoapsis) locations with respect to the central body.

* apoapsis: The point of a body's elliptical orbit about the system's centre of mass where the distance between the body and the centre of mass is at its maximum. Etymology: from *apo* (From the Ancient Greek prefix ἀπό- (apó-), from the preposition ἀπό (apó, "from, away from"),) + *apsis* (from Ancient Greek ἁψίς (hapsís) 'arch, vault'; pl. apsides is the farthest or nearest point in the orbit of a planetary body about its primary body.)

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