CCNY PHYS 35100 - Fall 2024 - Midterm Exam 1
Oct 9, 2024

Name Solutions

Instructions: There are 5 problems. Each part of each problem is worth the same amount. Please do your work on this test paper. There is extra paper attached to the back if you would like it for rough work, etc. When doing the problems, the more steps you justify, the more likely you are to receive full credit, in other words, please show your work. Show *how you know*, not just *that you know*. No external test aids may be used.

Some mathy things that might help

$$a^2 + b^2 = c^2 \neq (a+b)^2$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C$$

$$\int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2} + C$$

$$\int u\,dv = uv - \int v\,du$$

$$\sin^2\phi + \cos^2\phi = 1$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

1. Work out the first 3 terms for the Taylor series expansions of this function, around the point x = 0. (Terms that are =0 don't count in the tally) You don't need to write the summation series representation, just the polynomial.

a. $f(x) = xe^{-x}$

$f(0) = 0$

$f'(x) = e^{-x} - xe^{-x}$

$f'(0) = 1$

$f''(x) = -e^{-x} - (e^{-x} - xe^{-x})$

$= -2e^{-x} + xe^{-x}$

$f''(0) = -2$

$f'''(x) = 2e^{-x} + (e^{-x} - xe^{-x})$

$f'''(0) = 3$

$f(x) \approx 0 + \frac{1}{1}(x-0) + \frac{-2}{2!}(x-a)^2 + \frac{3}{3!}(x-a)^3$

$\left[\ \approx x - x^2 + \frac{x^3}{2!} + \dots\ \right]$

2. Back to the roots

A particle with mass m is traveling with speed v_0. At $t = 0$, it begins to be slowed down by a velocity dependant force given by $f = -av^{1/2}$, where a is a constant.

a. Set up Newton's second law for this system and determine the units [SI] of the constant a.

$$\sum F = m\ddot{x} = -a v^{1/2}$$

$$\frac{kg\, m}{s^2} = a \frac{m^{1/2}}{s^{1/2}} \quad \therefore \text{ a has to be } \frac{kg\, m^{1/2}}{s^{3/2}}$$

b. Solve the differential equation of motion to obtain a value of the distance traveled before coming to rest. (express in terms of v_0 and other relevant constants.)

$$m\dot{v} = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m\frac{dv}{dx}v = -av^{1/2}$$

$$\int_0^x -\frac{a}{m}\,dx = -\frac{a}{m}x$$

$$\int_{v_0}^{0} \frac{v}{v^{1/2}}\,dv = \int_{v_0}^{0} v^{1/2}\,dv = \frac{2}{3}v^{3/2}\Big]_{v_0}^{0} = -\frac{2}{3}v_0^{3/2}$$

$$\therefore -\frac{a}{m}x = -\frac{2}{3}v_0^{3/2}$$

$$\therefore x = \frac{2m}{3a}v_0^{3/2}$$

c. How long will it take to come to rest? (express t in terms of v_0 and other relevant constants.)

$$m\frac{dv}{dt} = -av^{1/2}$$

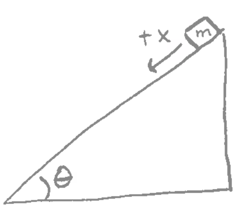
$$\int_{v_0}^{0} \frac{1}{\sqrt{v}}\,dv = -\int_0^t \frac{a}{m}\,dt \rightarrow 2v^{1/2}\Big]_{v_0}^{0} = -2v_0^{1/2} = -\frac{a}{m}t$$

$$\therefore t = \frac{2m}{a}\sqrt{v_0}$$

3. Every physics test needs a ramp

A mass m is let go at the top of a frictionless ramp. While there is no friction between the mass and the ramp, there is a quadratic dependant drag force that acts on the mass, and depends on the usual things (you can call that constant c).

a. Draw the system, indicate clearly your coordinates, and write out Newton's second law.



$$\sum F = m\ddot{x} = mg\sin\theta - cv^2$$

b. Find the terminal velocity of the mass, if one exists.

at $V_{term} \rightarrow \ddot{x} = 0$

$$\therefore 0 = mg\sin\theta - cv_{term}^2$$

$$\therefore V_{term}^2 = \frac{mg\sin\theta}{c} \rightarrow V_{term} = \sqrt{\frac{mg\sin\theta}{c}}$$

c. How much time does it take to reach a terminal velocity?

$$m\frac{dv}{dt} = mg\sin\theta - cv^2$$

$$\frac{m}{c}\frac{dv}{dt} = \frac{mg\sin\theta}{c} - v^2 = V_{term}^2 - v^2$$

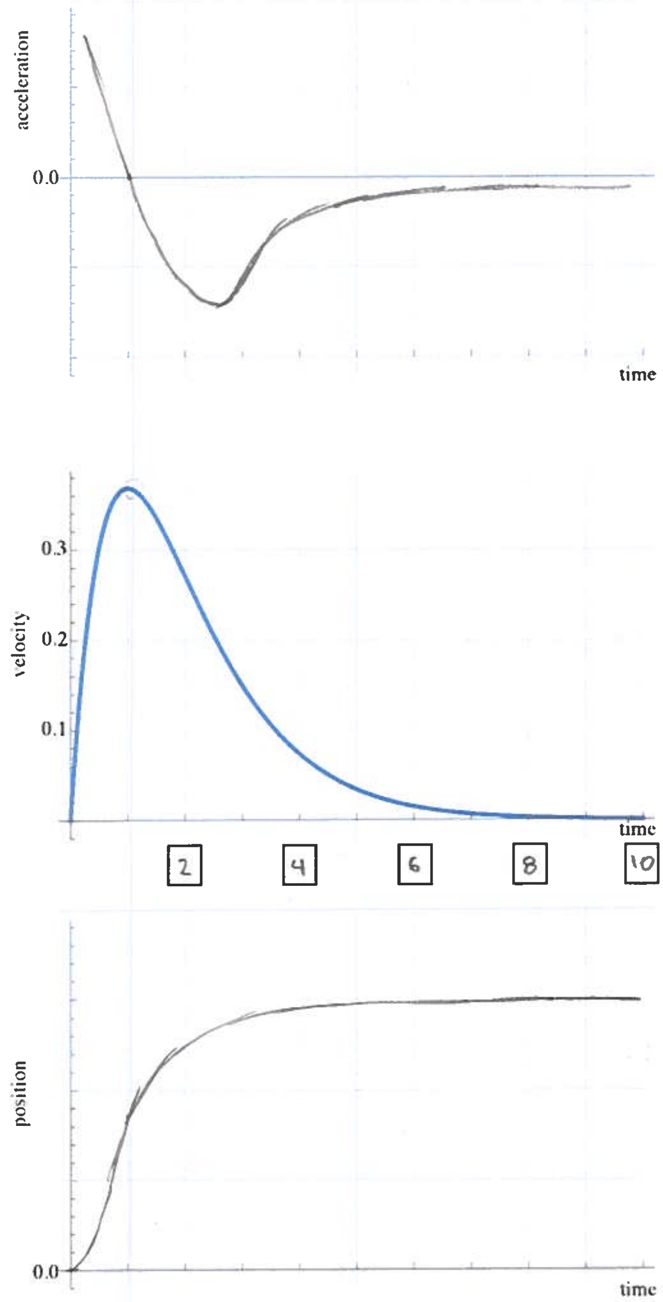
$$\rightarrow \int_0^{V_{term}} \frac{dv}{V_{term}^2 - v^2} = \int \frac{c}{m}\,dt = \frac{c}{m}t$$

↓ from front page

$$\frac{1}{V_{term}}\tanh^{-1}\left(\frac{V}{V_{term}}\right)\Big]_0^{V_{term}} = \frac{1}{V_{term}}\tanh^{-1}\left(\frac{V_{term}}{V_{term}}\right) = \frac{1}{V_{term}}\tanh^{-1}(1) = \text{uh oh.}$$

$$t = \infty$$

4. Velocity-Measure-O-Matic™



Here are 3 plots. In the middle, we have the velocity of an object as measured by our Velocity-Measure-O-Matic™. Through some fancy curve fitting, we found it followed a very simple function of time:

$$v(t) = t\,e^{-t}$$

(There should be some constants in the above to make the units nice, but let's just pretend they are all = 1)

a. Find the time when the velocity is a maximum and use that to fill in the empty boxes on the horizontal time axis (in seconds).

$V_{max} \rightarrow \frac{dv}{dt} = 0 \rightarrow \frac{dv}{dt} = e^{-t} - te^{-t}$

$0 = e^{-t} - te^{-t}$

$0 = 1 - t \Rightarrow t = 1$

b. Find the position as a function of time analytically using the above velocity function.

$\frac{dx}{dt} = v = te^{-t}$

$\int_0^x dx = \int_0^t te^{-t}\,dt$ } Integrate by parts

$\int u\,dv = uv - \int v\,du$

let $u = t$

$dv = e^{-t}dt$

$\therefore v = -e^{-t}$

$\frac{du}{dt} = 1$

$du = dt$

$\rightarrow \int te^{-t}dt = t(-e^{-t}) - \int(-e^{-t})dt$

$= t(-e^{-t}) - e^{-t} + 1$

$= -e^{-t}(t+1) + 1$

$\therefore x(t) = -e^{-t}(t+1) + 1 + x_0$

c. Make a qualitative sketch of the position vs time on the lowest plot. You can assume it starts at $x_0 = 0$.

d. Figure out the acceleration as a function of time analytically.

$a = \frac{dv}{dt} = e^{-t} - te^{-t}$

$= e^{-t}(1-t)$

e. Sketch that as well on the upper most graph.

5. I'm Loosing It

Our next system is a lightly damped, 1 dimensional simple oscillator (spring+mass). There is a spring force given by $-kx$ and a resistive force proportional to the velocity: $f_{\text{damp}} = -bv$. The object subject to these forces has a mass m.

a. Write out Newton's second law for this system.

$\sum F = m\ddot{x} = -kx - bv = -kx - b\dot{x}$

b. The kinetic energy is $\frac{mv^2}{2}$. Derive the potential energy using the relationship between force and potential.

Only conservative force is spring: $-kx$ $\quad U = -\int F\,dx = \frac{1}{2}kx^2$

c. In terms of the various constants, what is the total mechanical energy at the time $t = 0$ (assuming we release the oscillator from rest with an amplitude A .

at $t=0$, $v=0$, so M.E. $= \frac{1}{2}kx^2 + 0 \Rightarrow \frac{1}{2}kA^2$

d. Does the total mechanical energy change over time? (Yes or no)

e. Show that the rate of energy loss (that's another way of saying find dE/dt) is given by vf_{damp}

$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$\frac{dE}{dt} = \frac{1}{2}m\,2\dot{x}\ddot{x} + \frac{1}{2}k\,2x\dot{x}$

$= m\dot{x}\ddot{x} + kx\dot{x} \rightarrow (m\ddot{x} + kx)\dot{x}$

but from 2nd law: $m\ddot{x} + kx = -bv$, so

$\frac{dE}{dt} = -bv \cdot v = f_{\text{damp}}\,v$ ✓

f. The position of the oscillator was found in class to be: $x(t) = Ae^{-t/\tau}\cos(\omega_d t + \phi)$. Here, $\tau = 2m/b$ and ω_d is the damped frequency. What is the velocity as a function of time for this oscillator?

$v = \frac{dx}{dt} = A\left[\left(\frac{-1}{\tau}\right)e^{-t/\tau}\cos(\omega_d t + \varphi) + e^{-t/\tau}(-\sin(\omega_d t + \varphi))\omega_d\right]$

$= -Ae^{-t/\tau}\left(\frac{1}{\tau}\cos(\omega_d t + \varphi) + \omega_d\sin(\omega_d t + \varphi)\right)$

at $t=0$ — g. If the initial conditions are x_0 and v_0, figure out the quantity A.

$x_0 = A\cos(\varphi)$

$v_0 = -A\left(\frac{1}{\tau}\cos(\varphi) + \omega_d\sin(\varphi)\right)$

$\therefore \cos(\varphi) = \frac{x_0}{A}$ $\quad$ use $\sin(\varphi) = \sqrt{1-\cos^2(\varphi)}$

$v_0 = -A\left(\frac{1}{\tau}\frac{x_0}{A} + \omega_d\sqrt{1-\frac{x_0^2}{A^2}}\right)$

$v_0 = -\frac{x_0}{\tau} - A\omega_d\sqrt{1-\frac{x_0^2}{A^2}}$

$\left(v_0 + \frac{x_0}{\tau}\right)^2 = A^2\omega_d^2\left(1 - \frac{x_0^2}{A^2}\right)$

$\left(v_0 + \frac{x_0}{\tau}\right)^2 = A^2\omega_d^2 - \omega_d^2(x_0^2)$

$A^2 = \left(\left(v_0 + \frac{x_0}{\tau}\right)^2 + \omega_d^2 x_0^2\right)\frac{1}{\omega_d^2}$

$A = \sqrt{x_0^2 + \frac{\left(v_0 + \frac{x_0}{\tau}\right)^2}{\omega_d^2}}$

h. (is for hard) Use that $v(t)$ and our $x(t)$ to find, in the case of very light damping, i.e. $\omega_0\tau \Rightarrow \infty$, a nice neat expression for $E(t)$. (You can neglect any terms with a $1/\tau$ in them to help simplify so that it only has k, A, and τ as the parameters)

$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$

$= \frac{1}{2}mA^2e^{-2t/\tau}\left(\frac{1}{\tau}\cos(\omega_d t + \varphi) + \omega_d\sin(\omega_d t + \varphi)\right)^2 + \frac{1}{2}kA^2e^{-2t/\tau}\cos^2(\omega_d t + \varphi))$

anything with $1/\tau$ is small $\quad$ since $\omega_d \approx \omega_0 \approx \sqrt{\frac{k}{m}}$

$= \frac{1}{2}mA^2e^{-2t/\tau}\omega_d^2\sin^2(\omega t + \varphi) + \frac{1}{2}kA^2e^{-2t/\tau}\cos^2(\omega t + \varphi) = \frac{1}{2}kA^2e^{-2t/\tau} = E(t)$

also use $\sin^2(\,) + \cos^2(\,) = 1$