NY PHYS 35100 - Fall 2023 - Midterm 1 Name: Solutions

Instructor: J. Hedberg

Oct 16, 2023

Instructions: There are 5 problems. Each part of each problem is worth the same amount. Please do your work on this test paper. There is extra paper available if you would like it. When doing the problems, the more steps you justify, the more likely you are to receive full credit, in other words, please show your work. Show *how you know*, not just *that you know*. No external test aids may be used.

1. Leave me alone Mr. Taylor

Using the Taylor series definition:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

work out the first 4 terms for the series expansion of this natural log

$$f(x) = \ln(1-x)$$

for a point around 0.

$f(0) = \ln(1) = 0$

$f'(a) = \frac{1}{1-x}(-1)$

$f'(0) = -1$

$f''(a) = \frac{-1}{(1-x)^2}(-1)(-1)$

$f''(0) = -1$

$f'''(a) = \frac{-1}{(1-x)^3}(-1)(-2)$

$f'''(0) = -2$

$f''''(a) = \frac{-1}{(1-x)^4}(-1)(-2\times 3)$

$f''''(0) = -6$

$\therefore\ f(x) \approx 0 + (-1)x + \frac{(-1)x^2}{2!} + \frac{(-2)x^3}{3!} + \frac{(-6)x^4}{4!}$

$\therefore\ \ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$

2. Slow down, and come to a complete halt

A sphere of mass m is moving at speed v_0 with no forces acting. At time $t = 0$, a resistive force starts to act with a strength proportional to the square of the particles' velocity and some constant c that depends on the cross sectional area of the sphere and the nature of the medium it is traveling in.

a. Write a statement of Newton's second law describing the particle for times $t > 0$.

Solve the resulting differential equation and obtain expressions for

b. the velocity as a function of time, $v(t)$

c. the position as a function of time, $x(t)$

d. Considering the $v(t)$ and $x(t)$ functions you have, show (or explain) why the particle never actually stops moving.

e. Try to resolve this apparent conflict with reality, that is, objects obviously do slow down and stop when moving through a resistive medium, so what's wrong with our model?

a) $\sum F_x = m\ddot{x} = -cv^2$

b) $\frac{dv}{dt} = -\frac{c}{m}v^2$

$$\int_{v_0}^{v} \frac{1}{v^2}\,dv = -\frac{c}{m}\int_0^t dt$$

$$\left[\frac{-1}{v}\right]_{v_0}^{v} = -\frac{c}{m}t$$

$$-\frac{1}{v} + \frac{1}{v_0} = -\frac{c}{m}t$$

$$\frac{1}{v} = \frac{1}{v_0} + \frac{c}{m}t$$

$$= \frac{m}{mv_0} + \frac{v_0 c}{mv_0}t$$

$$\frac{1}{v} = \frac{m + v_0 c t}{m v_0}$$

$$v = \frac{m v_0}{m + v_0 c t}$$

$$\boxed{v(t) = v_0\left(\frac{1}{1 + \frac{cv_0}{m}t}\right)}$$

c) $v = \frac{dx}{dt}$

$$\therefore \int_{x_0}^{x} dx = \int_0^t v\,dt$$

$$x - x_0 = \int_0^t v_0\left(\frac{1}{1 + \frac{cv_0}{m}t}\right)dt$$

$$= v_0\left[\frac{m}{cv_0}\left(\ln\left(1 + \frac{v_0 c}{m}t\right)\right)\right]_0^t$$

$$\boxed{x(t) = x_0 + \frac{m}{c}\ln\left(1 + \frac{v_0 c}{m}t\right)}$$

d) for $v(t)$ unless $t = \infty$

$\frac{1}{1 + \theta t}$ will always be > 0

meaning it keeps moving.

simarly $\ln(1 + \theta t)$ keep increasing too.

e) when v becomes small enough, then linear drag will dominate.

Linear drag $x(t)$ has an asymptotic approach to a max value.

3. Through the middle

On the surface of a regular rocky type planet with mass m_p and radius R_p and density ρ, like Earth or Mars, the *attractive* force of gravity acting on another object of mass m_1, near the surface, would be given by:

$$F_G = G\frac{m_1 m_p}{{R_p}^2} \tag{3.1}$$

Now, imagine a tunnel drilled from the north pole to the south pole of this planet. If the object is allowed to travel through the tunnel, the force of Gravity will change. It can be shown that the strength of the force will be given by a similar equation, except that the m_p will not be the total mass of the planet, but only the portion of its mass contained within a sphere with radius r where r is the distance between the object and the center, and the R_p will not be the radius of the planet, but rather the same distance r between the object and center. (This was proved by Newton)

a. Convert this statement to an equation for the gravitational force inside the tunnel for an object. It should depend on m_1, ρ, G, and r, and have some numerical constants.

b. Show that if the object is allowed to released at the entrance to the tunnel (i.e. just at the surface), we would expect to see a oscillatory behavior for its position.

c. What would the natural frequency and period be of its oscillations be? (i.e. find ω_0 and T) for this simple harmonic oscillator (SHO) in terms of the quantities given.

d. If the particle where to be given an initial velocity v_0 as it was thrown down the tunnel (starting from R_p), find expressions for A and ϕ in terms of these initial conditions and ω_0 in the SHO equation:

$$r = A\cos(\omega_0 t + \phi) \tag{3.2}$$

a)

On surface

$$F_G = -G\frac{m_1 m_p}{R_p^2}$$

inside

$$F_G = -\frac{G m_1\left(\frac{4}{3}\pi r^3 \rho\right)}{r^2}$$

mass of inner sphere:

$$\frac{4}{3}\pi r^3 \cdot \rho = m_{in}$$

$$\boxed{F_G = -\left(\frac{4}{3}\pi\rho G\right)m_1 r}$$

↓ now linear in r

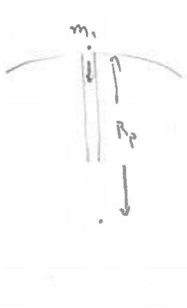
b) $\sum F_r = m\ddot{r} = -\left(\frac{4}{3}\pi\rho G\right)m_1 r$

↓ rename as H

$\ddot{r} = -Hr$ } S.H.O because

if

$$\left.\begin{aligned} r &= A\cos(\omega_0 t + \varphi) \\ \dot{r} &= -A\omega_0\sin(\omega_0 t + \varphi) \\ \ddot{r} &= -A\omega_0^2\cos(\omega_0 t + \varphi) \end{aligned}\right\} \text{cos is a solution ✓}$$



c) since $\ddot{r} = -Hr$

$$-A\omega_0^2\cos(\omega_0 t + \varphi) = -HA\cos(\omega_0 t + \varphi)$$

$$\omega_0^2 = H$$

$$\therefore\ \omega_0 = \sqrt{\tfrac{4}{3}\pi\rho G}$$

$$\text{\& } T = \frac{1}{f} = \frac{2\pi}{\omega_0} = \boxed{\frac{2\pi}{\sqrt{4/3\,\pi\rho G}}}$$

d) at $t = 0$,

$$R_p = A\cos(\varphi)$$

$$-v_0 = -A\omega_0\sin(\varphi)$$

$$\therefore\ \frac{\sin(\varphi)}{\cos(\varphi)} = \tan(\varphi) = \frac{v_0/\omega_0}{R_p} \Rightarrow \varphi = \tan^{-1}\left(\frac{v_0}{\omega_0 R_p}\right)$$

$$\text{\& } R_p^2 + \frac{v_0^2}{\omega_0^2} = A^2\cos^2(\varphi) + A^2\sin(\varphi)$$

$$= A^2(1)$$

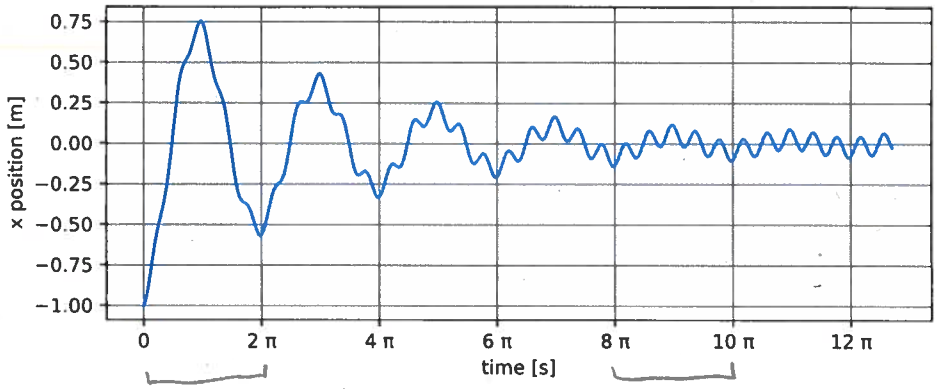
$$\therefore\ A = \sqrt{R_p^2 + \frac{v_0^2}{\omega_0^2}}$$

4. Interpret a plot

Here is a plot of the position vs. time for a damped driven spring-mass oscillator that is the solution to the following force equation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \beta v + \frac{F_0}{m}\sin(\omega t + \phi) \tag{1}$$

At $t = 0$: $v_0 = 0$ and $x_0 = -1$.



$2\pi\,s$ = Period   5 oscillations in 2π s

Respond to the following questions (For the Multiple Choice questions, select the best choice and write that letter in the corresponding box.

a. Would you say this oscillator is: $\boxed{A}$

A. Lightly (under)Damped
B. Critically Damped
C. Overdamped

b. Which choice is closest to the natural frequency (ω_0) of the oscillator? $\boxed{B}$

A. 0.5 rad/sec
B. 1.0 rad/sec
C. 2.5 rad/sec
D. 5.0 rad/sec
E. 10.0 rad/sec

$T = 2\pi$

$f = \frac{1}{2\pi}$

$\omega_0 = 2\pi \times f = 2\pi \times \frac{1}{2\pi} = 1$

c. Which choice is closest to the driving frequency (ω) of the oscillator? $\boxed{D}$

A. 0.5 rad/sec
B. 1.0 rad/sec
C. 2.5 rad/sec
D. 5.0 rad/sec
E. 10.0 rad/sec

$T = \frac{2\pi}{5}$

$f = \frac{5}{2\pi}$

$\omega_0 = 2\pi \times \frac{5}{2\pi} = 5$

d. Using your results from question 1 of the exam, do a quick, back of the envelope type, numerical calculation to obtain a value for the damping coefficient β.

amplitude $\sim e^{-t\beta}$

Looking at the graph, the amplitude drops by 1/2 after ~1 oscillation, or 2π s

$\therefore\ 0.5 = e^{-t\beta} = e^{-2\pi\beta}$

$\ln(.5) = -2\pi\beta = \ln(1-.5)$

from Taylor: $\ln(1-.5) \approx -.5 - \frac{.5^2}{2}$

$= -.5 - \frac{.25}{2} = -.5 - 0.125 - \theta^3$

$= -.6125 - \theta^3$

thus: $-2\pi\beta \approx -.613$

but $2\pi \approx 2 \times 3.14 = 6.28$

therefore $\beta = \frac{.613}{6.28} \approx .1$

5. Bye Bye Balloon

A spherical Helium filled ballon is released from rest from the surface of the Earth. It begins to rise due to the buoyancy force acting, but experiences a linear drag force contrary to its velocity (i.e. $f_d = -bv$). (For simplicity, you can assume that the skin of the balloon is very light weight and can be considered negligible compared to the helium gas inside the balloon)

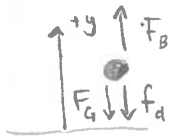
a. Draw a Free Body Diagram, a coordinate system (with up being +y) and write the Sums of Forces equations (2nd law) for this situation.

b. Solve the differential equation to obtain an expression for velocity as a function of time.

c. Find an expression for the terminal velocity of the balloon as it rises.

d. Show how this terminal velocity depends on the density of the Helium gas.

a)



$$\Sigma F = +F_B - F_G - f_d$$

$$F_B \to \frac{4}{3}\pi r^3 \rho_{air}\, g \quad \text{or} \quad \frac{m_B \rho_{air}}{\rho_B} g \qquad F_G \to m_B g \qquad f_d \to bv$$

$$\Sigma F = m\ddot{y} = \underbrace{m_B g\left(\frac{\rho_{air}}{\rho_B}\right) - m_B g}_{g' = \left(\frac{\rho_{air}}{\rho_B} - 1\right) g} - bv$$

b)

$$\ddot{y} = \frac{dv}{dt} = g' - \frac{b}{m_B} v$$

$$\int_0^v \frac{dv}{g' - \frac{b}{m_B} v} = \int_0^t dt$$

$$-\frac{m}{b}\left(\ln\left(g' - \frac{b}{m_B} v\right)\right)\Big|_0^v = t$$

$$-\frac{m}{b}\left(\ln\left(g' - \frac{b}{m_B} v\right) - \ln g'\right) = t$$

$$-\frac{m}{b} \ln \frac{g' - \frac{b}{m_B} v}{g'} = t$$

$$g' - \frac{b}{m_B} v = g' e^{-b/m_B t}$$

$$v(t) = \frac{m_B}{b} g'\left(1 - e^{-b/m\, t}\right)$$

c) $v_{term} = v(t \to \infty)$

$$= \frac{m_B}{b} g'(1 - 0)$$

$$V_{term} = \frac{m_B}{b} g'$$

d) ρ_{He} is baked into g'

$$V_{term} = \frac{m_B}{b}\left(\frac{\rho_{air}}{\rho_B} - 1\right)$$

$\rho_B \to \rho_{ballon} \approx \rho_{Helium}$

but also in $m_B \sim V_b \rho_{He}$

$$V_{term} \propto \rho_{air} - \rho_{He}$$