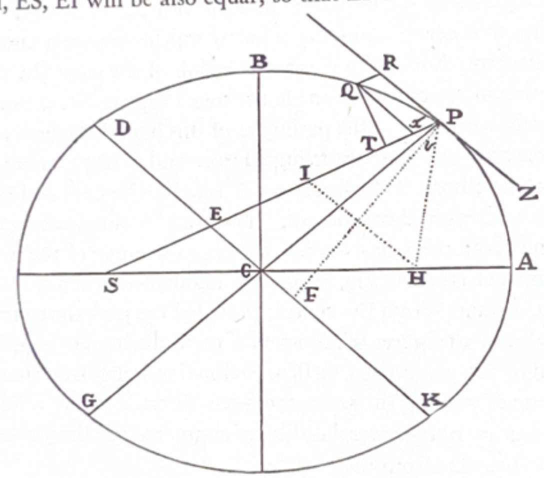
SECTION III

The motion of bodies in eccentric conic sections.

PROPOSITION XI. PROBLEM VI

If a body revolves in an ellipse; it is required to find the law of the centripetal force tending to the focus of the ellipse.

Let S be the focus of the ellipse. Draw SP cutting the diameter DK of the ellipse in E, and the ordinate Qv in x; and complete the parallelogram $QxPR$. It is evident that EP is equal to the greater semiaxis AC: for drawing HI from the other focus H of the ellipse parallel to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half-sum of PS, PI,



that is (because of the parallels HI, PR, and the equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis 2AC. Draw QT perpendicular to SP, and putting L for the principal latus rectum of the ellipse (or for $\frac{2BC^2}{AC}$), we shall have

$$L \cdot QR : L \cdot Pv = QR : Pv = PE : PC = AC : PC,$$

$$\text{also, } L \cdot Pv : Gv \cdot Pv = L : Gv, \text{ and, } Gv \cdot Pv : Qv^2 = PC^2 : CD^2.$$

By Cor. II, Lem. VII, when the points P and Q coincide, $Qv^2 = Qx^2$, and Qx^2 or $Qv^2 : QT^2 = EP^2 : PF^2 = CA^2 : PF^2$, and (by Lem. XII) $= CD^2 : CB^2$. Multiplying together corresponding terms of the four proportions, and simplifying, we shall have

$$L \cdot QR : QT^2 = AC \cdot L \cdot PC^2 \cdot CD^2 : PC \cdot Gv \cdot CD^2 \cdot CB^2 = 2PC : Gv,$$

since $AC \cdot L = 2BC^2$. But the points Q and P coinciding, 2PC and Gv are equal. And therefore the quantities $L \cdot QR$ and QT^2, proportional to these, will be also equal. Let those equals be multiplied by $\frac{SP^2}{QR}$, and $L \cdot SP^2$ will become equal to $\frac{SP^2 \cdot QT^2}{QR}$. And therefore (by Cor. I and V, Prop. VI) the centripetal force is inversely as $L \cdot SP^2$, that is, inversely as the square of the distance SP. Q.E.I.

The same otherwise.

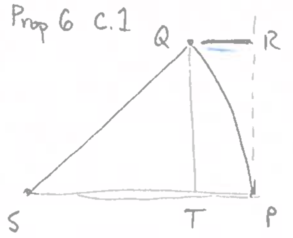
Since the force tending to the centre of the ellipse, by which the body P may revolve in that ellipse, is (by Cor. I, Prop. X) as the distance CP of the body from the centre C of the ellipse, let CE be drawn parallel to the tangent PR of the ellipse; and the force by which the same body P may revolve about any other point S of the ellipse, if CE and PS intersect in E, will be as $\frac{PE^3}{SP^2}$ (by Cor. III, Prop. VII); that is, if the point S is the focus of the ellipse, and therefore PE be given as SP^2 reciprocally. Q.E.I.

With the same brevity with which we reduced the fifth Problem to the parabola, and hyperbola, we might do the like here; but because of the dignity of the Problem and its use in what follows, I shall confirm the other cases by particular demonstrations.

Adapted from Newton's Principia, the Central Argument
Densmore, Green Lion Press, 2003

Start

Prop 6 C.1



$$\frac{1}{\text{force}} \propto \frac{SP^2 \times QT^2}{QR}$$

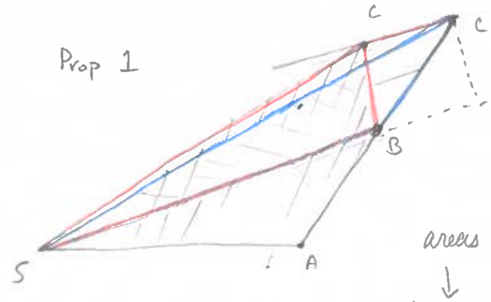
QR ~ distance deflected

SP·QT ~ time

$\frac{\text{distance}}{\text{time}^2}$ is like acceleration

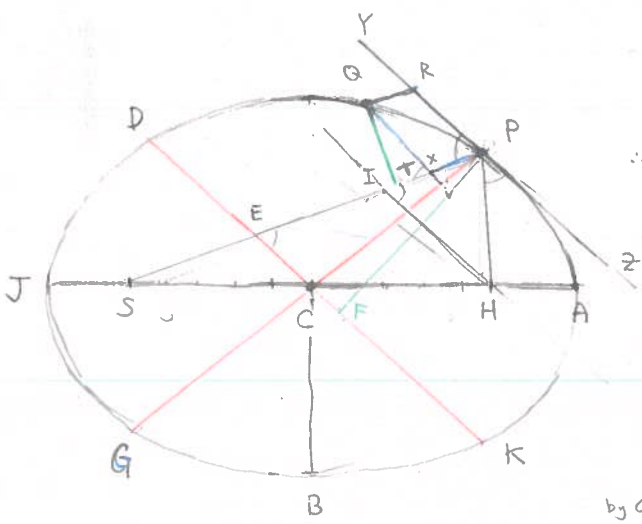
$f \sim a$

Prop 1



areas $\propto$ time
↓
$\frac{1}{2}$(base × height)

Prop 11: If a body revolve in an ellipse, it is required to find the law of centripetal force tending to the focus of the ellipse.



evident: EP = AC

by construction IH // PZ // DK

$\therefore \angle YPS = \angle PIH$

$\angle HPZ = \angle PHI$

$\angle YPS = \angle HPZ$ (Apollonius III 48)

$\therefore \angle PIH = \angle PHI$

$\therefore IP = PH$

$\frac{SE}{SI} = \frac{SC}{CH}$ (Euclid VI.2)

but SC = CH (Apollonius III 45)

so SE = EI

by C. EP = EI + PI

thus EP = SE + PI

$2EP = \underbrace{EP + SE}_{} + PI$

$2EP = SP + PI$ (Apollonius III.52)

$EP = \frac{SP + PI}{2} = \frac{SP + HP}{2} = \frac{AJ}{2} = AC$

$\therefore EP = AC$

Proportions:

1: Latus Rectum = $L = \frac{2BC^2}{AC}$

$\frac{Px}{Pv} = \frac{PE}{PC}$

$\frac{QR}{Pv} = \frac{AC}{PC}$

1. $\boxed{\frac{L \times QR}{L \times Pv} = \frac{AC}{PC}}$

2. $\frac{L}{Gv} = \frac{L}{Gv}$

$\boxed{\frac{L \times Pv}{Gv \times Pv} = \frac{L}{Gv}}$

Apollonius I.21

3. $\frac{DC^2}{Qv^2} = \frac{GC \times CP}{Gv \times vP}$

but GC = CP

$\frac{DC^2}{Qv^2} = \frac{CP^2}{Gv \times vP}$

$\boxed{\frac{Gv \times vP}{Qv^2} = \frac{CP^2}{DC^2}}$

4. PF ⊥ DK

QT ⊥ SP

$\angle PFE = \angle QTx$

$\angle PEF = \angle QxT$

$\therefore \triangle PEF \sim \triangle QxT$

$\therefore \frac{Qx}{QT} = \frac{PE}{PF} = \frac{CA}{PF}$

$\frac{Qx}{QT} = \frac{CD}{CB}$

in the limit: $Qx \to Qv$

$\frac{Qv}{QT} = \frac{CD}{CB}$ or $\boxed{\frac{Qv^2}{QT^2} = \frac{CD^2}{CB^2}}$

But parallelogram $\overset{\text{Maj / Min}}{AJ, 2CB} = DK, PG$ } Lemma 12

1/4's are equal to $\therefore \underset{CA}{CA} \times CB = \underset{CD}{CD} \times PF$ (areas of ▱)

$$\frac{L \times QR}{\boxed{L \times Pv}} = \frac{AC}{PC}$$

$$\frac{\boxed{L \times Pv}}{\boxed{Pv \cdot Gv}} = \frac{L}{Gv}$$

$$\frac{Gv \times vP}{\boxed{Qv^2}} = \frac{CP^2}{DC^2}$$

$$\frac{Qv^2}{QT^2} = \frac{CD^2}{CB^2}$$

$$\frac{L \times QR}{QT^2} = \frac{AC \times L \times PC^2 \times CD^2}{PC \times Gv \times DC^2 \times CB^2}$$

$$= \frac{AC \times L \times PC}{Gv \times CB^2} \qquad \text{but } L = \frac{2BC^2}{AC}$$

$$\frac{L \times QR}{QT^2} = \frac{2BC^2 \times PC}{Gv \times CB^2} = \frac{2PC}{Gv}$$

in the limit $2PC \overset{\lim}{=} GV$, so

$$\frac{L \times QR}{QT^2} = 1 \quad \text{or} \quad L \times QR = QT^2$$

$\therefore$ multiply both sides by SP^2/QR

$$\frac{L \times QR \times SP^2}{QR} = \frac{QT^2 \times SP^2}{QR}$$

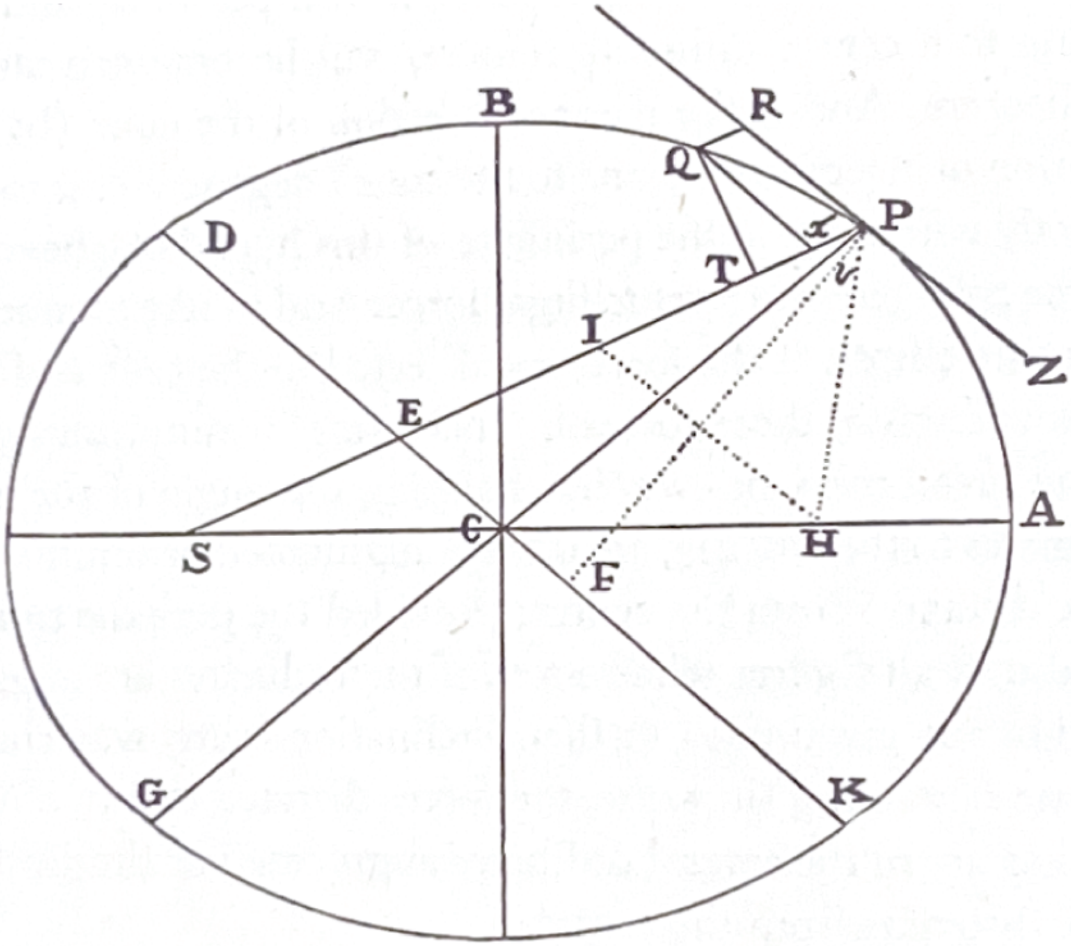
$$\therefore \quad L \times SP^2 = \frac{QT^2 \times SP^2}{QR}$$

but, due to Prop 6, C1,

$$\frac{QT^2 \times SP^2}{QR} \text{ is } \propto \frac{1}{\text{force}}$$

so

$$\text{force} \propto \frac{1}{L \times SP^2}$$, L is a constant (Latus Rectum) of the ellipse.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