On motion in a resisting medium: A historical perspective

William W. Hackborn[a)]
Mathematical Sciences, Augustana Campus, University of Alberta, Camrose, Alberta, Canada T4V 2R3

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This paper examines, compares, and contrasts ideas about motion, especially the motion of a body in a resisting medium, proposed by Galileo, Newton, and Tartaglia, the author of the first text on exterior ballistics, within the context of the Aristotelian philosophy prevalent when these scholars developed their ideas. This historical perspective offers insights on the emergence of a scientific paradigm for motion, particularly with respect to the challenge of incorporating into this paradigm the role played by the medium. *© 2016 American Association of Physics Teachers.*
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I. INTRODUCTION

> We bring forward a brand new science concerning a very old subject.
>
> There is perhaps nothing in nature older than MOTION, about which volumes neither few nor small have been written by philosophers; yet I find many essentials of it that are worth knowing which have not even been remarked, let alone demonstrated.[1]

So begins the Third Day of Galileo's dialogue, *Two New Sciences*. The subject indicated is the motion of objects dropped or thrown, which, in light of its importance for our hunting ancestors, is very old indeed. It is also ancient in a philosophical sense: Aristotle's *Physics*, said by Martin Heidegger to be "the fundamental book in western philosophy,"[2] and Aristotle's *On the Heavens*[3] are the primary works referenced by Galileo, who, like many philosophers before him, wrestled with Aristotle's ideas. Aristotle's *Physics* deals with nature broadly defined, and the questions it addresses are very different from those addressed by the physics familiar to us. On first encountering it, Kuhn wondered, "How could [Aristotle] have said about [motion] so many apparently absurd things?," but this and similar encounters eventually led Kuhn to his idea of shifting scientific paradigms.[4] Aristotle defines motion (Greek *kinesis*) as "the actuality of what exists in potency, as such,"[5] and he identifies three kinds: "the one of quality, the one of amount, and the one according to place."[6] Galileo's analysis of the last kind, *local motion* (Latin *motus localis*) as he calls it, in *Two New Sciences* differs greatly from Aristotle's (and Galileo's own youthful) analysis of it. One of the factors that complicates any analysis of local motion is the (almost) inevitable presence of a resisting medium. Like Aristotle and Galileo, Tartaglia, whose seminal work on ballistics made him one of Galileo's most influential precursors, and Newton, whose principles of natural philosophy vastly extended the reach of Galileo's kinematics, dealt with the resisted motion of a body in creative, perceptive, and powerful ways.

II. TARTAGLIA

Niccolò Tartaglia (1500–1557) acquired his surname, which means "stammer," from a speech defect caused by wounds inflicted by French soldiers. He overcame this and other challenges to make substantial contributions to renaissance science, including the first Italian translations of Euclid and Archimedes, and the first solution of certain kinds of cubic equations.[7] Books I and II of Tartaglia's *Nova Scientia*[8] (1537) present a mathematical theory of exterior ballistics that remained popular well beyond Galileo's death a century later[9] and is largely consistent with Aristotle's ideas on local motion. In Aristotle's *Physics*, local motion is either "by nature" or "by violence": natural motion is the movement of something by its own agency (like an animal walking) or the vertical movement of "heavy" things downward and "light" things upward to "their proper places"; violent motion results from the action of something else (like a spear thrown by a hand).[10] Furthermore, local motion "is either in a circle, or on a straight line, or mixed."[11] Tartaglia's theory is founded on this "logic of contraries":[12] natural versus violent, and circular versus rectilinear, motion.

Nova Scientia adopts a Euclidean approach, deducing propositions from definitions, suppositions, and axioms. Book I begins by defining a "uniformly heavy" body as one "which, according to the weight of the material and its shape, is apt not to suffer noticeable resistance from the air" and proceeds to define other terms applicable to such a body, including natural and violent motion.[13] His reference to "shape" here shows clearly that Tartaglia, like Aristotle,[14] is aware of its effect on air drag. The meaning of "noticeable," on the other hand, is less certain. It may refer only to what is noticeable at the level of ordinary experience (e.g., the dropping of an object from a tower), but if taken literally it invalidates his projection trajectory for a uniformly heavy body, as noted below. Two (of the six) propositions in Book I, asserting that uniformly heavy bodies accelerate in natural motion and decelerate in violent motion, follow easily from suppositions associating a body moving at greater speed with greater impact (on an object struck) and axioms associating greater impact with a body falling naturally from greater height or moving violently through less distance.[15] Another, asserting the impossibility of "mixed natural and violent motion," is deduced from the inconsistency of simultaneous deceleration and acceleration.[16]

Book II of *Nova Scientia* begins with definitions and suppositions describing the trajectory of a uniformly heavy projectile: the main claim is that any non-vertical trajectory not terminated prematurely consists of two lines joined smoothly by the arc of a circle, one line being the line of projection (above or below the horizontal) and the other line vertical[16] (see Fig. 1). Note that, although such a projectile "is apt not to suffer noticeable resistance from the air,"[13] this trajectory differs noticeably from the direction-invariant trajectory

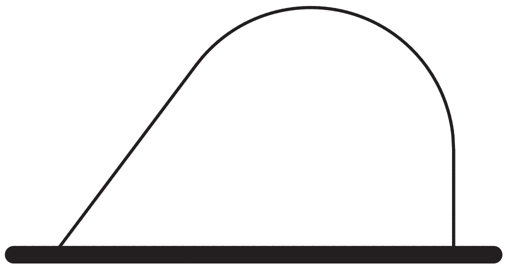


Fig. 1. Tartaglia's trajectory: a line at an acute angle to the horizontal and a vertical line joined smoothly by an arc of a circle.

with fore-aft symmetry that must occur in the absence of air resistance. The first two parts of the trajectory, the line of projection and the circular arc, represent violent motion, and the last part, the vertical line, describes natural fall. Hence, from propositions in Book I, the projectile decelerates from the moment of its launch until it reaches the end of the circular arc and then accelerates as it falls straight down. The first six (of the nine) propositions in Book II are simple geometric consequences of this form of trajectory. Proposition VII states that the violent portions of all trajectories with the same projection angle are geometrically similar, but its proof rests on an assumption equivalent to the proposition itself.[17] Proposition VIII employs a peculiar "physical argument" to assert that a projectile attains its maximum range when the launch angle is 45°.[18] Finally, Proposition IX claims that the initial straight part of the trajectory of a body projected at 45° is about four times longer than that of the same body projected horizontally using "the same motive power."[19]

As with natural and violent motion, Aristotle drew a sharp distinction between celestial and terrestrial phenomena. For him, "only the movement of heavenly objects composed of *aether* had a perfect regularity that allowed their motion to be described mathematically," while "mathematics, the consideration of abstract quantities, was inappropriate to [terrestrial] physics" (but see Ref. 23); this makes Tartaglia "an important liminal figure in the transformation of Aristotelian natural science … because he … blurred Aristotelian philosophical categories" and adopted "new standards of mathematical precision" for terrestrial motion, "cannonshot" specifically; the "physics of projectile motion was a … peripheral problem in the framework of Aristotle's natural philosophy, but cannon warfare gave the problem … new urgency," which allowed Tartaglia's ideas to undermine Aristotle's authority.[20] Moreover, Tartaglia used his geometric description of a cannonball's trajectory in a non-literal, abstract way—a modern, mathematical model. For example, he remarked that "no [non-vertical] violent trajectory … can have any part that is perfectly straight" owing to the ball's weight "which … draws it toward the center of the world," but he took the part of the trajectory "which is insensibly curved to be straight, and that which is evidently curved… to be part of … a circle."[21] Although this three-part trajectory is strange, we will see in Sec. IV that it bears some resemblance to the trajectory of a cannonball. This, alongside the general harmony of Tartaglia's ideas and those of Aristotle, the simplicity of his model, the consistency of Propositions VII and VIII in Book II with Galileo's parabolic trajectory, the fact that he consulted with bombardiers, and the influence of Venice's military commanders whom he addressed,[8] may account for the persistence of his new science.

III. GALILEO

Galileo Galilei (1564–1642) was occupied with motion for much of his career. In early manuscripts written while teaching at the University of Pisa (1589–1592) and published posthumously as *De Motu*,[22] Galileo often contends with Aristotle on the subject. In his *Physics*, Aristotle claims the speed of a body in natural vertical motion is proportional both to how "subtle" or "easily divisible" is the medium through which it moves (i.e., speed is inversely proportional to the medium's density, presumably, or viscosity) and to its "excess" of "heaviness or lightness."[23] Although "excess" here is ambiguous, its meaning is clarified by Aristotle's own examples, such as his claim that "a mass of gold or lead … is quicker in proportion to its size."[24] Galileo affirms that a body's "essential heaviness or lightness," which he equates to the body's specific weight (i.e., the weight of a given volume of the material making up the body), causes its downward or upward motion and offers arguments that the speed of motion varies as the absolute difference of the specific weight of the body and that of the medium; hence two bodies of the same material (such as gold) but different total weight move naturally through the same medium with the same speed.[25] This *weight-buoyancy* theory of natural motion conflates speed with force (in Newton's sense) and is almost identical to a doctrine of G. Benedetti announced in 1553; Benedetti, unlike Galileo, cites (in arguing for this doctrine) Book I of Archimedes' *On Floating Bodies*,[26] for which the likely source is its translation in Tartaglia's 1551 treatise on raising sunken ships; in fact, a remark in this treatise on the speed of bodies sinking in water possibly inspired the ideas of both Benedetti and Galileo.[27] Notably, Tartaglia taught Benedetti and, it is widely believed, O. Ricci, mathematics teacher to the young Galileo.[7]

Aristotle's *Physics* rarely addresses projectile motion; its major claim is that the violent motion of a projectile is sustained, temporarily, by the medium after contact with the projector ceases.[28] This claim was debated by later scholars. Hipparchus (ca. 2nd century B.C.E.) proposed that projection upwards impresses a self-exhausting force on the projectile that sustains its upward motion for a time and impedes its eventual downward motion, and J. Buridan (ca. 14th century C.E.) conceived *impetus*, an impressed force that persists unchanged unless the projectile's motion is opposed or promoted by other factors, such as the medium's resistance or the projectile's weight; Buridan explained the acceleration of a falling body using increments to its impetus caused by its own weight.[29] *De Motu* posits a theory of impressed force like that of Hipparchus: the force impressed by throwing a body "upward" or "downward" reduces its "heaviness" or "lightness," respectively, but not "its natural weight," and a dropped body has a "force impressed on it equal to its own weight;" the impressed force "will finally be lost and the natural weight resumed," so "acceleration will cease."[30] Hence, acceleration is temporary in Galileo's weight-buoyancy theory on the speed of natural motion. A later chapter of *De Motu*[31] that links projection angle with how far a projectile moves "on a straight line" indicates that Galileo subscribed at this time to both Tartaglia's image of a projectile's trajectory and Aristotle's logic of contraries.[32]

Two New Sciences (1638) was composed by Galileo under house arrest late in his life but is based on research done decades earlier. Written as a dialogue among three friends meeting in the Venetian arsenal over a period of four days, the science of material strength is the main topic discussed in its first two days, the science of motion in its last two days. The three participants in the dialogue are Salviati, Galileo's spokesman, Simplicio, an Aristotelian philosopher, and Sagredo, a knowledgeable layman who often voices the views held by Galileo earlier in his career.[33] Galileo's ideas on motion in *Two New Sciences* differ from those in *De Motu* in important ways. In the former, for example, Sagredo explains the motion of a "heavy body hurled upwards" in terms of "the force impressed upon it by the thrower" much as *De Motu* does, but Salviati dismisses this and other explanations of the "cause of the acceleration of natural motion" as "fantasies" and advises that "[f]or the present, … our Author [Galileo] … want[s] us to investigate and demonstrate some attributes of … [naturally accelerated] motion … (whatever be the cause of its acceleration)."[34] As this suggests, *Two New Sciences* advocates kinematic principles investigated by "actual measurement" and mathematically demonstrated as opposed to "hidden causes" uncovered by "subtle verbal reasoning" common in the Aristotelian tradition.[33] Nevertheless, like *De Motu*, *Two New Sciences* often uses causal explanations.[35] One cause that Galileo, like Tartaglia, considers is the "impact" of, say, "a sledge fall[ing] on a pole" whose "effect" is measured by the depth the sledge drives the pole into the ground: this "effect … [is] greater … according as the height is greater from which the impact is made; that is, according as the speed of the striking body is greater," says Salviati.[36] Salviati provides another causal explanation when he argues, as Tartaglia does, that the "curvature of the line of the horizontal[ly fired] projectile seems to derive from two forces, of which one (that of the projector) drives it [the projectile] horizontally, while the other (that of its own heaviness) draws it straight down," likening these moving forces to the static forces acting on a tightly stretched rope.[37] In any case, unlike *De Motu*, many of the results in *Two New Sciences* are supported by carefully executed experiments, although much about these experiments (as reconstructed from Galileo's working notes) are hotly debated.[35] In one reconstruction based on his notes from 1602–1604, Galileo found the time-squared law for distance fallen in a roundabout way (using his rule relating the period of a small-amplitude pendulum to its length) by finding "the length of the pendulum which swings through a small arc to the vertical while a body falls [a given distance];"[33] another, based on his notes from 1604–1609, has Galileo discovering the parabolic trajectory of a projectile using a ball rolling off a table, a discovery delayed until Galileo discarded the idea that speed of free fall varies as distance fallen.[34,38]

The Third Day of *Two New Sciences* opens with a Latin treatise, *On Local Motion*, which includes a brief introduction and two theory-laden books, *On Equable Motion*[39] and *On Naturally Accelerated Motion*.[40] The rest of the Third Day consists of Italian dialogue that intersperses the second book of the treatise and clarifies it, often with Salviati schooling the other two speakers (e.g., see Ref. 34). This second book assumes naturally accelerated motion is "uniformly and continually accelerated" in that "in any equal times, equal additions of swiftness are added on," and proceeds to demonstrate Proposition I, which asserts that the time in which a uniformly accelerated body moves some distance from rest equals the time in which a body moves equably the same distance at half the final speed of the accelerated body, and Proposition II, the time-squared rule for distance moved by a uniformly accelerated body starting from rest, which Galileo states as "the spaces run through in any times whatever are to each other as the duplicate ratio of their times."[41] Most of the remaining 36 propositions in The Third Day involve times of descent on inclined planes, and these rely on an earlier postulate that the speed acquired by a body moving down an inclined plane with "all obstacles and impediments removed" depends only on the plane's vertical height.[42] Proposition IV, for example, asserts that the ratio of descent times from rest on two inclined planes varies inversely as the square root of the ratio of their heights.[43] Another, Proposition VI, deals with inclined planes whose cross-sections are chords of a vertical circle with one end at the circle's highest or lowest point: Galileo proves descent times from rest on all such planes are equal, a surprising and elegant result.

The Fourth Day of *Two New Sciences* contains the third book, *On the Motion of Projectiles*,[44] of the Latin treatise begun in the Third Day, interspersed with Italian dialogue. Proposition I describes the trajectory of a body projected horizontally with "motion compounded from equable horizontal and from naturally accelerated downward" motions as "semiparabolic." Its proof assumes the two components of motion are independent and relies on Galileo's inertial principle, "that whatever degree of speed is found in the moveable, this is by its nature indelibly impressed on it when external causes of acceleration or retardation are removed, which occurs only on the horizontal plane."[45] In Propositions II–IV, Galileo develops a geometric method for determining the speed of a projectile at any point on its trajectory. This method and two more propositions culminate in Proposition VII, which asserts that of all semiparabolic trajectories with the same "amplitude" (horizontal range), the projectile on the trajectory whose "altitude" (initial height) is half its amplitude reaches the end of its trajectory with the least speed; the end of this trajectory is sloped at 45°, and Galileo, by cleverly reversing the direction of motion, concludes that "maximum projection" for a given initial speed is achieved with "elevation of half a right angle."[46]

Although the treatise *On Local Motion* presupposes the absence of all impediments to movement, *Two New Sciences* is set sharply apart from *De Motu* by a long discussion about air resistance.[47] The discussion begins with Simplicio, the Aristotelian, objecting to Salviati's views on two grounds: the non-existence of a horizontal plane (on which motion persists equably) equidistant at all points from the center of the Earth, and "the impediment of the medium" that destroys the equability of horizontal motion and the uniform acceleration of vertical. Salviati deals quickly with the first, comparing an artillery shot of four miles with Earth's radius to show that a horizontal plane is a valid abstraction, but acknowledges that the second objection is "more considerable." Air resistance, he says, "is incapable of being subjected to firm rules" due to "the infinitely many ways that the shapes of the moveables vary, and their heaviness, and their speeds," that "to deal with such matters scientifically, it is necessary to abstract from them," and that for projectiles "of heavy material and spherical shape, and … [others] of less heavy material and cylindrical shape, as are arrows, launched by slings or bows, the deviations from exact parabolic paths will be quite insensible." Salviati notes that a falling body "ought to go on accelerating" but that eventually "the impediment of the air … will reduce it to … equable motion" and that "this equilibration will occur more quickly … as the moveable shall be less heavy." This is a causal explanation: "air exercises its force," he says, and he describes some experiments that reveal aspects of this force. He uses one experiment to suggest that the air drag on a body is proportional to its speed, and another to argue that "the force of gunpowder" can shoot a ball at "supernatural" speed (i.e., exceeding its "terminal speed" of fall in air).[48] Salviati concedes "some deformation" from a parabolic path occurs for a cannonball fired at supernatural speed but claims no such deformation occurs "in practicable operations" involving "mortars charged with but little powder." The final propositions, XII–XIV, in The Fourth Day are associated with ballistic tables, two of which give the amplitudes and altitudes of the semi-parabolic paths of cannonballs fired at the same speed for each degree of projection angle.[49]

IV. NEWTON

The study of motion is central to the *Principia* of Isaac Newton (1642–1727) and a key application for the infinitesimal calculus he devised. An early draft of *Principia* was titled *De Motu Corporum*, and this remained the title of the first two books of the published text. The third and final book, *De Mundi Systemate*, investigates motion of various kinds in the solar system.[50] Galileo's work significantly influenced that of Newton,[51] who acknowledges, following the statement of his laws of motion, that Galileo used the first two of those laws to find the time-squared rule for falling bodies and the parabolic path of a projectile.[52] While forces lie near the periphery of *Two New Sciences*, abstract mathematical descriptions of forces are fundamental to the methodology of *Principia*.[53] Book 2 of *Principia* deals with motion in resisting fluids. Newton's interest in such motion stems partly from the Aristotelian tradition he inherited. That he wrestled with that tradition as an undergraduate is shown by an early notebook (ca. 1664) in which he rejects both Aristotle's notion about the surrounding medium and the medieval idea of impetus as causes for the persistence of a projectile's motion.[54] Moreover, Newton remarks that he has "illustrated" the "principles of philosophy" set forth in Books 1 and 2 using topics "that seem to be the most fundamental for philosophy," including "resistance of bodies."[55] However, as Newton discredits Descartes' vortex theory of celestial motion in the last (of nine) sections of Book 2 and in the General Scholium at the end of *Principia*, many take this as his prime reason for writing Book 2.[56]

The first section of Book 2 investigates motion under resistance proportional to speed, the relationship suggested by Galileo.[48] Extending earlier propositions on resisted horizontal and vertical motion, Proposition 4 of Book 2 and its corollaries address the motion of a projectile.[57] In modern notation, the problem considered in this proposition is described by

$$\frac{dx}{dt} = u\,, \quad \frac{dy}{dt} = v\,, \quad \frac{du}{dt} = -\frac{f(s)}{s}\,u\,,$$
$$\frac{dv}{dt} = -g - \frac{f(s)}{s}\,v\,, \tag{1}$$

where (x, y) is the position vector of the projectile, (u, v) its velocity, $s = \sqrt{u^2 + v^2}$ its speed, $f(s) = ks$ (with $k > 0$ constant) the magnitude of the resisting force on it per unit mass, and g its natural downward acceleration in the fluid medium (due to both gravity and the buoyancy caused by its displacement of fluid). Taking the initial conditions as $(x, y) = (0, 0)$ and $(u, v) = (\alpha, \beta)$, where $\alpha > 0$ and β are constants, solving Eqs. (1) for x and y, and eliminating t from the result yields

$$y = \left(\beta + \frac{g}{k}\right)\frac{x}{\alpha} + \frac{g}{k^2}\log\left(1 - \frac{kx}{\alpha}\right). \tag{2}$$

Instead of deducing this Cartesian equation, Newton constructed the projectile's path geometrically, representing physical quantities as lines, ratios, and areas. He represents the initial velocity of a body thrown at point D as line DP, the vertical component of the initial "resistance of the medium … [relative] to the force of gravity" as ratio $DA{:}AC$ (where A is a point chosen on the horizontal axis DC to make this so, and PC is the vertical axis), and time elapsed as the area of region $DRTG$ bounded below by segment DR of DC, above by segment GT of a rectangular hyperbola with asymptotes DC and PC, and on its sides by vertical segments DG and RT (with R chosen arbitrarily on DC); he gives instructions for finding r on extended segment RT so that "curved line $DraF$" is the projectile's path "which point r traces out" (with a the highest point of the path, segment Aa vertical, the path intersecting DC at F "and afterward always approaching the asymptote PC"). In Corollary 2 of Proposition 4, he remarks that "it is easy to draw curve $DraF$ with the help of a table of logarithms." Like Galileo, Newton wrote mostly in the style of Euclid and Archimedes, but sometimes he used mixed ratios, modern algebraic expressions, and relationships that hold only in limiting cases (e.g., of "innumerable rectangles").[57]

Newton remarks that the hypothesis of resistance proportional to speed "belongs more to mathematics than to nature" and that, in "mediums wholly lacking in rigidity" (i.e., rarefied fluids in which viscous forces are negligible compared to inertia[58]), both the "quantity of the medium … disturbed" and the momentum imparted to it by a moving body per unit time are proportional to the body's speed; thus, "by the second and third laws of motion," he argues, "resistances encountered by bodies are as the squares of the velocities."[57] Section 2 of Book 2 examines motion resisted in this way, culminating in Proposition 10, which tackles an inverse problem: determine, given a curve, how the medium's density must vary with position so that a projectile follows that curve as it moves under gravity and drag, assumed to vary jointly as the density and the speed squared; this inverse problem typifies Newton's method of using motions to determine "causes and effects."[53] He tries four sample trajectories: a semi-circle, a parabola, a hyperbola (with oblique and vertical asymptotes to its ascending and descending parts, respectively), and a generalized hyperbola (with similar asymptotes).[59] From the results, he infers that the projectile's trajectory in a uniformly dense medium "approaches closer to these hyperbolas than to a parabola" and is itself "of a hyperbolic kind" (i.e., the trajectory has oblique and vertical asymptotes, like the hyperbolas he considers). Finally, he states eight rules that describe hyperbolas (conic or generalized) approximating this trajectory under various (e.g., initial) conditions. It is curious that a semi-circle is one of the sample trajectories Newton considers, as one piece of Tartaglia's trajectory is part of a semi-circle. More remarkably, Newton's "hyperbolic kind" of trajectory, with its asymptotes, bears notable similarity to that of Tartaglia shown in Fig. 1.

A "serious error" in Newton's analysis for the semicircular trajectory in Proposition 10 of the first edition of *Principia* had unexpected consequences. Johann Bernoulli discovered the error and communicated it to Newton through his nephew, Nikolaus Bernoulli, who was visiting London in September 1712. Newton then tailored a long paste-up correction for the second (1713) edition, already printed, without citing Bernoulli's help, further fueling the calculus priority dispute raging between mathematicians on opposite sides of the English channel at this time. Bernoulli charged that Newton did not understand higher derivatives (fluxions) and their connection to infinite series, used extensively in Proposition 10.[60] Eventually, responding to a 1717 challenge from Oxford professor John Keill to "[f]ind the curve which a projectile describes" subject to gravity and fluid drag varying as the square of its speed,[61] Johann Bernoulli found an expression for this curve in the case of drag varying as an arbitrary power of speed:[62] $Z = \int (a^2 + z^2)^{n-1/2}\, dz$, $y = \int aZ^{-1/n}\, dz$, $x = \int zZ^{-1/n}\, dz$, where the meanings of y and x are the reverse of those in Eq. (1), the resistance of the medium varies as the projectile's speed to the power $2n \in \mathbb{R}$, and a is unspecified. A brief derivation follows. Letting $w = v/u$ and taking $f(s) = ks^{2n}$, Eq. (1) implies

$$\frac{dw}{dt} = -\frac{g}{u}, \quad \frac{du}{dw} = \frac{f(s)}{gs}\, u^2 = \frac{k}{g} u^{2n+1}(1 + w^2)^{n-1/2}. \qquad (3)$$

Using the initial values $(u, v) = (\alpha, \beta)$ and $\alpha > 0$ in Eq. (3) yields[63]

$$u = \left[\frac{1}{\alpha^{2n}} - \frac{2nk}{g}\int_{\beta/\alpha}^{w} (1 + w^2)^{n-1/2}\, dw\right]^{-1/2n}. \qquad (4)$$

Also, $dt = -(u/g)\, dw$ from Eq. (3), while $dx = u\, dt$ and $dy = uw\, dt$, so[64]

$$t = -\frac{1}{g}\int_{\beta/\alpha}^{w} u\, dw, \quad x = -\frac{1}{g}\int_{\beta/\alpha}^{w} u^2\, dw,$$
$$y = -\frac{1}{g}\int_{\beta/\alpha}^{w} u^2 w\, dw. \qquad (5)$$

Bernoulli's solution can be reconciled with Eqs. (4) and (5), which express time elapsed and position as functions of $w = v/u$, the slope of the projectile's trajectory. A local analysis near the fixed point for (u, v) shows $w \to -\infty$ monotonically as $t \to \infty$ if $n > 0$.[63] For $n = 1$ and many other values of n, Eq. (4) leads to an elementary expression for u in terms of w.

Section 3 of Book 2 examines motion "resisted partly in the ratio of the velocity and partly in the squared ratio of the velocity." The purpose behind this section becomes clear when Newton states at the end of Section 3 that "resistance encountered by spherical bodies in fluids arises partly from the tenacity, partly from the friction, and partly from the density of the medium,"[65] equivalent to representing fluid drag as $F_D = a + bs + cs^2$ for a body moving at speed s, where a is the fluid's tenacity, b its coefficient of viscous drag (due to its internal friction or rigidity), and c its coefficient of inertial drag (due to its density).[58] Sections 6 and 7,[66] essentially, are intended to provide a theoretical and experimental basis for determining values for the constants a, b, and c. Section 6 addresses the motion of pendulums in resisting fluids; Newton hoped to measure the separate components of resistance by observing the decay of pendulum motion, but in this he was disappointed, later concluding that the sloshing of the fluid induced by the pendulum opposed its motion and increased drag significantly. The experiments in Section 7 were more successful; these involved timing the vertical fall of spherical bodies in water and air, the most dramatic being the dropping of glass balls (filled with air or mercury) and inflated hog bladders in London's St. Paul's Cathedral. In Section 7, Newton also derives expressions for the inertial drag (in effect, expressions for the coefficient c above) of a sphere in an inviscid, incompressible fluid of two kinds, "rarefied" and "continuous." Using his expression for a "continuous" medium and his solution for the vertical fall of a body resisted as s^2 (Proposition 9), he then calculates the theoretical outcomes corresponding to his observations of vertical fall and finds good agreement between theory and experiment, apparently allowing the possibility of measuring the other drag components from the difference between theoretical and observed results.

In hindsight, Newton's representation of the drag force is flawed.[53,58] The expression for F_D above, with constant a, b, and c, presumes the viscous and inertial components of drag are independent when, in fact, they are intimately linked via the character of the flow around the body, which is determined at subsonic speeds for a body of given shape and surface attributes by a dimensionless parameter, the *Reynolds number* Re.[67] Researchers now express the drag on a given body in a *Newtonian* fluid of density ρ by $F_D = (1/2)C_D \rho A s^2$, where A is the body's *frontal area* and C_D is a dimensionless *drag coefficient* that varies with the Reynolds number, as determined by experiments on a scale model, and more so with the Mach number M at transonic and supersonic speeds.[68] As yet, C_D cannot be obtained from theory, and Newton's expressions derived in Section 7 for the inertial drag (which, ignoring viscous drag, amount to $C_D = 1$ for a sphere and $C_D = 2$ for a cylinder moving parallel to its axis in a "rarefied" medium, and $C_D = 0.5$ for a body of any shape in a "continuous" medium) are wrong;[58] the theoretical drag (equivalent to $C_D = 0.5$) Newton used for his comparisons with observations just happened (for the Re values in his vertical fall experiments) to agree fairly well with the actual drag. In fact, Newton's assumptions of an incompressible, inviscid fluid that slips tangentially on a solid boundary lead paradoxically (as Jean d'Alembert showed using theory decades later) to zero drag on a body of any shape.[53] On the other hand, Newton's inertial drag (cs^2 in effect) closely matches the modern expression, $F_D = (1/2)C_D \rho A s^2$, since C_D is roughly constant over wide intervals of Re and M for many bodies,[67,68] and Section 7 confirms that it varies jointly as fluid density and frontal area.[66] Moreover, Newton's vertical fall experiments yield C_D values well within modern ranges. Sadly, historians have mostly ignored these pioneering experiments and Book 2 in general.[58]

We can now address Galileo's claim that drag is negligible for "practicable" mortar shots (see Fig. 2).[48] Although linear drag holds in air only for tiny particles at low speeds ($\text{Re} \lesssim 1$), Fig. 2 includes it for comparison; the terminal speed in Fig. 2 was calculated from that of an iron shot, assuming a cannonball with the same density and drag coefficient.[67] The launch speed, about half the speed of sound (near which C_D is roughly constant), and launch angle in Fig. 2 are appropriate for a mortar raining fire on a fortress.

V. SUMMARY AND CONCLUSIONS

We have followed a thread of thinking, from Aristotle to Newton and beyond, about motion and the medium in which it occurs. Aristotle realized that the medium plays a dual role: it gives a body both resistance to motion and buoyancy. With respect to the former, he knew it was affected by the body's shape,[14] and he knew the latter (which exists in the layers of water and air held to our planet by gravity) is determined by the body and the medium: wood, he writes, is heavier than air but lighter than water.[69] Within the context of Aristotle's ideas, Tartaglia created a plausible mathematical model of projectile motion, but the restriction of his theory to "uniformly heavy" bodies[16] suggests he was possibly unaware that air drag causes the asymmetry in a body's trajectory. In *De Motu*, Galileo clarifies Aristotle's concepts of heaviness and lightness using the notion of specific weight, but he wrongly identifies the speed of a body with the difference between its specific weight and that of the medium, and he ignores the medium's resistance, as if it plays no role.[25]

By contrast, the treatment of air resistance in *Two New Sciences* is strikingly modern: the terminal speed of a falling body is seen to result from a balance of gravity and drag, and air resistance is said to be "incapable of being subjected to firm rules,"[47] an intuition borne out by current research.[58] In its understanding of drag, its extensive use of mathematics, its empirical basis, and its attempt to renounce hidden causes (such as the self-exhausting impressed force used in *De Motu*), some have seen *Two New Sciences* as a paradigm shift in thinking about motion.[70] The emergence of the classical view of motion, however, was more an evolutionary process than a sudden shift effected by one text or one person. The seeds of this view were planted by occasional musings about local motion within Aristotle's much larger vision of nature as a whole; they were watered by Archimedes, medieval scholars, Tartaglia, Galileo, and other scientists; and they were brought to fruition by those who continued and expanded these earlier efforts,[71] Newton among them. Book 2 of his *Principia* truly created a new science of fluid resistance, but he was a bit too optimistic about the firmness of the rules governing it. Given its importance for physics and real life, it's surprising that oversimplified rules for fluid resistance sometimes creep into physics textbooks. Here is one example from my first undergraduate physics text, a sentence that inspired my interest in fluid drag: "Over a wide range of values of [velocity] v, this fluid frictional resistance is well described by the following formula: $R(v) = Av + Bv^2$ where A and B are constants."[72]

Galileo and Newton both helped to bring motion into the "mathematical tradition" comprising such "classical sciences" as astronomy and optics,[73] and to create in the 18th century "a near-perfect … fusion between mathematics and mechanics, to the immeasurable advantage of both, and of all physics and technology to come."[74] A nice example of this fusion is Bernoulli's solution for resisted motion, which, beyond its mathematical beauty, bestowed harsh military power. Using tables computed from this solution for quadratic drag ($n = 1$), Leonhard Euler devised a method[75] used for mortar fire up to World War II;[76] for high-speed projectiles, others computed trajectories as a sequence of arcs associated with different n values[68] using a piecewise smooth "drag function."[77] This, however, is another story.

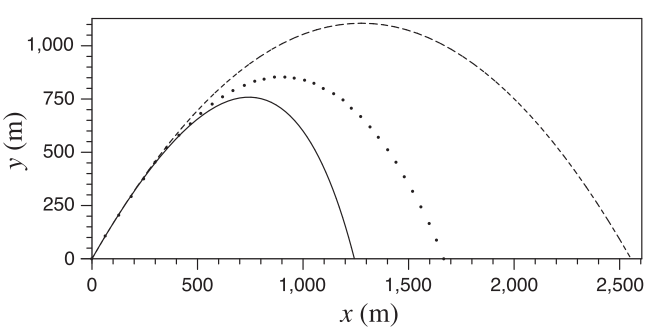


Fig. 2. Trajectories of a 58-kg cannonball (diameter 24 cm) fired with an initial speed of 170 m/s at a 60° angle, subject to linear drag (solid), quadratic drag (dotted), and no drag (dashed). The results for linear and quadratic drag were found using Eq. (2) and by numerically solving Eq. (1) with $f(s) = ks^2$; in both cases, k was chosen to give a terminal speed of 205 m/s.

[a)]Electronic mail: hackborn@ualberta.ca

[1]Galileo Galilei, *Two New Sciences*, translated with a new introduction and notes by Stillman Drake, 2nd ed. (Wall and Emerson, Toronto, 1989), p. 147.

[2]Aristotle, *Physics, or Natural Hearing*, translated and introduced by Glen Coughlin (St. Augustine's Press, South Bend, IN, 2005), p. ix.

[3]Aristotle, *On the Heavens*, translated by J. L. Stocks, in *The Complete Works of Aristotle*, edited by J. Barnes (Princeton U.P., Princeton, NJ, 1984), Vol. 1, pp. 447–511. *On the Heavens* deals in more depth with some topics introduced in Aristotle's *Physics*, and it contrasts motion in the celestial and terrestrial realms.

[4]Thomas S. Kuhn, *The Essential Tension: Selected Studies in Scientific Tradition and Change* (University of Chicago Press, Chicago, 1977), pp. x–xiii. Kuhn's encounter with Aristotle's *Physics* occurred in 1947 when, as a doctoral student in physics, he was asked to give a series of talks on the origins of mechanics.

[5]Reference 2, 201a10–201a14. Bekker numbers, the standard method for indexing and citing Aristotle's writings, are used in this paper. They refer to page, column, and line numbers in A. I. Bekker's 1831 edition of Aristotle's complete works.

[6]Reference 2, 225b5–225b9.

[7]Stillman Drake and I. E. Drabkin, *Mechanics in Sixteenth-Century Italy: Selections from Tartaglia, Benedetti, Guido Ubaldo, and Galileo* (The University of Wisconsin Press, Madison, WI, 1969), pp. 16–26.

[8]Reference 7, pp. 61–97.

[9]A. Rupert Hall, *Ballistics in the Seventeenth Century* (Cambridge U.P., Cambridge, 1952), p. 36.

[10]Reference 2, 254b7–256a3.

[11]Reference 2, 265a13–265a15. In *On the Heavens*, Aristotle further argues that matter in the celestial realm is composed of "aether," a substance that "moves in a circle … naturally" and differs from the four terrestrial elements: earth, water, air, and fire (see Ref. 3, 269b18–270b25).

[12]Peter Damerow, Gideon Freundenthal, Peter McLaughlin, and Jürgen Renn, *Exploring the Limits of Preclassical Mechanics* (Springer-Verlag, New York, 1992), pp. 78–91.

[13]Reference 7, pp. 70–72.

[14]Reference 2, 216a15–216a19: "it divides either by its shape or by the inclination which … the projectile has."

[15]Reference 7, pp. 73–78. Tartaglia's recognition of acceleration in free fall is consistent with that of Aristotle, who states that things "borne on a straight line" are "borne faster" as they are "borne further away from … resting" (Ref. 2, 265b10–265b14). Tartaglia associates acceleration loosely with a body moving "more swiftly the more it shall depart from its beginning or the more it shall approach its end" (p. 74). This differs sharply from Galileo's precise notion of uniform acceleration associated with time (not distance) of fall.

[16]Reference 7, pp. 80–86.

[17]Reference 7, pp. 89–91: "when the straight parts of violent trajectories … correspond, their distances also correspond, otherwise it would be very contradictory."

[18]Reference 7, pp. 91–94: a cannonball shot horizontally "would complete its violent motion farther below the plane of the horizon than when elevated in any other direction" and, shot vertically, violent motion would "terminate higher … above the plane of the horizon than … at any other elevation;" thus, "there will be an elevation … to make this [violent motion] terminate precisely in the plane of the horizon," and this is "that elevation … midway between those … most contrary in results" (p. 93); Tartaglia then invokes Supposition IV in Book II: "the most distant effect … that can be made … in violent motion upon any plane … is [in] that … which terminates precisely in this plane" (p. 85).

[19]Reference 7, pp. 94–97. In the proof, Tartaglia takes "as an assumption … that the distance of the violent trajectory … elevated at 45 degrees above the horizon is about ten times the straight trajectory made along the horizontal plane" (p. 94). He uses this and Proposition VIII to obtain a quadratic equation for the ratio of the initial straight part of the trajectory of a body projected at 45° to that of the same body shot horizontally using the same power, and he finds this ratio to be $\sqrt{200} - 10$, roughly $4\frac{1}{7}$.

[20]Mary J. Henninger-Voss, "How the 'new science' of Cannons shook up the Aristotelian cosmos," J. Hist. Ideas **63**, 371–397 (2002).

[21]Reference 7, pp. 84–85.

[22]Galileo Galilei, *On Motion and On Mechanics*, comprising *De Motu* (ca. 1590), translated with introduction and notes by I. E. Drabkin, and *Le Meccaniche* (ca. 1600), translated with introduction and notes by Stillman Drake (The University of Wisconsin Press, Madison, WI, 1960), pp. 3–4.

[23]Reference 2, 215a25–216a20. Aristotle's explicit use of proportions (ratios) here shows that he is not averse to applying mathematics to some terrestrial phenomena.

[24]Reference 3, p 309b14.

[25]Reference 22, pp. 13–40.

[26]Reference 7, p. 149. Benedetti refers to *On Floating Bodies* within the argument (pp. 148–152) for this doctrine in his *Resolutio* (1553).

[27]Stillman Drake, *History of Free Fall: Aristotle to Galileo*, printed as an Appendix to Ref. 1, pp. 27–29.

[28]Reference 2, pp. 266b25–267b25. When "things thrown" are no longer "touched by the mover," they continue to move because "the air or the water moves, being divisible," as "things contiguous to each other" in a succession of motions, with successively decreasing "power for moving," initiated by the thrower and extending to the projectile, which eventually stops.

[29]Reference 27, pp. 8–9 and 18–19.

[30]Reference 22, pp. 76–105.

[31]Reference 22, pp. 110–114.

[32]Reference 12, pp. 144–147.

[33]Reference 1, pp. xiii–xxxv.

[34]Reference 1, pp. 159–160.

[35]Reference 12, pp. 126–264, takes a very careful look at the transition from *De Motu* to Two New Sciences, including the use of impressed force and the logic of contraries to explain projectile motion in the latter (pp. 251–262) and links from theory in *De Motu* to both the time-squared rule for distance fallen and the parabolic path of a projectile (pp. 149–158).

[36]Reference 1, p. 156. If Salviati's claim is understood in terms of ratios, as seems likely, then it agrees and disagrees with classical mechanics, in that an effect defined this way is determined by the striking body's ability to do work, which is proportional to its initial height (via potential energy) and to the *square* of its impact speed (via kinetic energy).

[37]Reference 1, p. 256.

[38]Reference 27, p. 54.

[39]Reference 1, pp. 147–153. *On Equable Motion* presents the principles of motion at constant speed, most of which date back to Aristotle's *Physics*, using propositions that set forth the proportions (ratios) associated with such motion (see Ref. 12, pp. 11–15).

[40]Reference 1, pp. 153–216.

[41]Reference 1, pp. 165–167. Proposition I follows from the *Medieval Mean Speed Theorem*, originated at Merton College, Oxford, in the 14th century (Ref. 27, pp. 17–18) and developed further in the geometrical "configuration of qualities" method of Nicole Oresme. Some essentials of this method were not passed on to the 17th century, but Galileo used geometric aspects of it in the proof of this proposition (see Ref. 12, pp. 15–19 and 228–230). Galileo's statement of Proposition II and others adheres to Euclidean proportion theory (*Elements*, Book V), which Galileo uses to compare continuous magnitudes of the same kind (see Ref. 33). Euclidean proportions, though less flexible than modern equations, allow physical laws to be stated without constants like g.

[42]Reference 1, p. 162. Salviati later (pp. 169–170) describes experiments involving a "wooden beam" and a "hard bronze ball." In classical mechanics, a solid ball has a constant acceleration of $(5/7)g\sin\theta$ on a plane inclined at angle θ when subject to just enough friction to make it roll without slipping and assuming all its potential energy is converted to kinetic energy, as any physics undergrad can confirm using moment of inertia.

[43]Note that *De Motu* also deals with motion on inclined planes, claiming that the ratio of descent speeds on two such planes varies inversely as the ratio of their lengths (Ref. 22, pp. 63–69).

[44]Reference 1, pp. 217–260.

[45]Reference 1, p. 197. Galileo's principles of horizontal inertia and the independence of horizontal and vertical motion are based on causal explanations and originate in *De Motu* (see Ref. 12, pp. 147 and 155).

[46]Reference 1, pp. 234–245. Galileo never justifies his use of direction reversal to derive a trajectory due to oblique projection from one due to horizontal projection; this gap may reflect his flawed attempts to deduce the former directly (see Ref. 12, pp. 205–213 and 241–242).

[47]Reference 1, pp. 223–229.

[48]Reference 1, pp. 226–229. The first of these two experiments involves "two equal lead balls" hanging by "two equal threads," both balls moved from the vertical, one "by 80 degrees or more," the other "by no more than four or five degrees," and released. Salviati claims the two pendulums remain in tandem for up to "hundreds" of oscillations, but Drake reports (footnote 12) that they vary by "one beat after about the first thirty." If the pendulums stayed in tandem and a pendulum's period did not increase with amplitude, Sagredo's conclusion that "speed … is itself both the cause and the measure of … resistance" would seem more valid. The second (thought) experiment invokes a "bullet [fired] from an arquebus vertically downward on a stone pavement," once from a great height and again from a few feet: Salviati believes the bullet fired from great height is "less flattened" because drag reduces its impact speed.

[49]Reference 1, pp. 249–255.

[50]Isaac Newton, *The Principia: Mathematical Principles of Natural Philosophy*, a New Translation by I. Bernard Cohen and Anne Whitman assisted by Julia Budenz, preceded by *A Guide to Newton's Principia* (pp. 1–370) by I. Bernard Cohen (University of California Press, Berkeley, 1999), pp. 11–13. Editions of *Principia* were published in 1687, 1713, and 1726.

[51]John Herivel, *The Background to Newton's Principia: A Study of Newton's Dynamical Researches in the Years 1664–84* (Oxford U.P., Oxford, 1965), pp. 35–41.

[52]Reference 50, p. 424.

[53]George E. Smith, "The methodology of the *Principia*," in *The Cambridge Companion to Newton*, edited by I. Bernard Cohen and George E. Smith (Cambridge U.P., Cambridge, 2002), pp. 138–173.

[54]Reference 51, p. 1.

[55]Reference 50, p. 793.

[56]Reference 50, pp. 164–165, 779–790, and 939–944.

[57]Reference 50, pp. 633–641.

[58]George E. Smith, "Another way of considering Book 2: Some achievements of Book 2," in Ref. 50, pp. 188–194. See also Ref. 50, pp. 161–167, for a general introduction to Book 2.

[59]Reference 50, pp. 655–669. On the generalized hyperbola, vertical distance from its oblique asymptote varies inversely as an arbitrary power—unity in the case of a conic hyperbola—of distance from its vertical asymptote.

[60]Michael Nauenberg, "Proposition 10, Book 2, in the *Principia*, revisited," Arch. Hist. Exact Sci. **65**, 567–587 (2011). This details Newton's analysis for Proposition 10 and the conceptual error in it (which persisted somewhat in his "corrected" version). See also Ref. 50, pp. 168–171.

[61]Reference 9, pp. 154–156.

[62]Johann Bernoulli, "Responsio ad nonneminis provocationem, ejusque solutio quaestionis ipsi ab eodem propositae, de invenienda linea curva quam describit projectile in medio resistente," Acta Eruditorum **38**, 216–226 (1719).

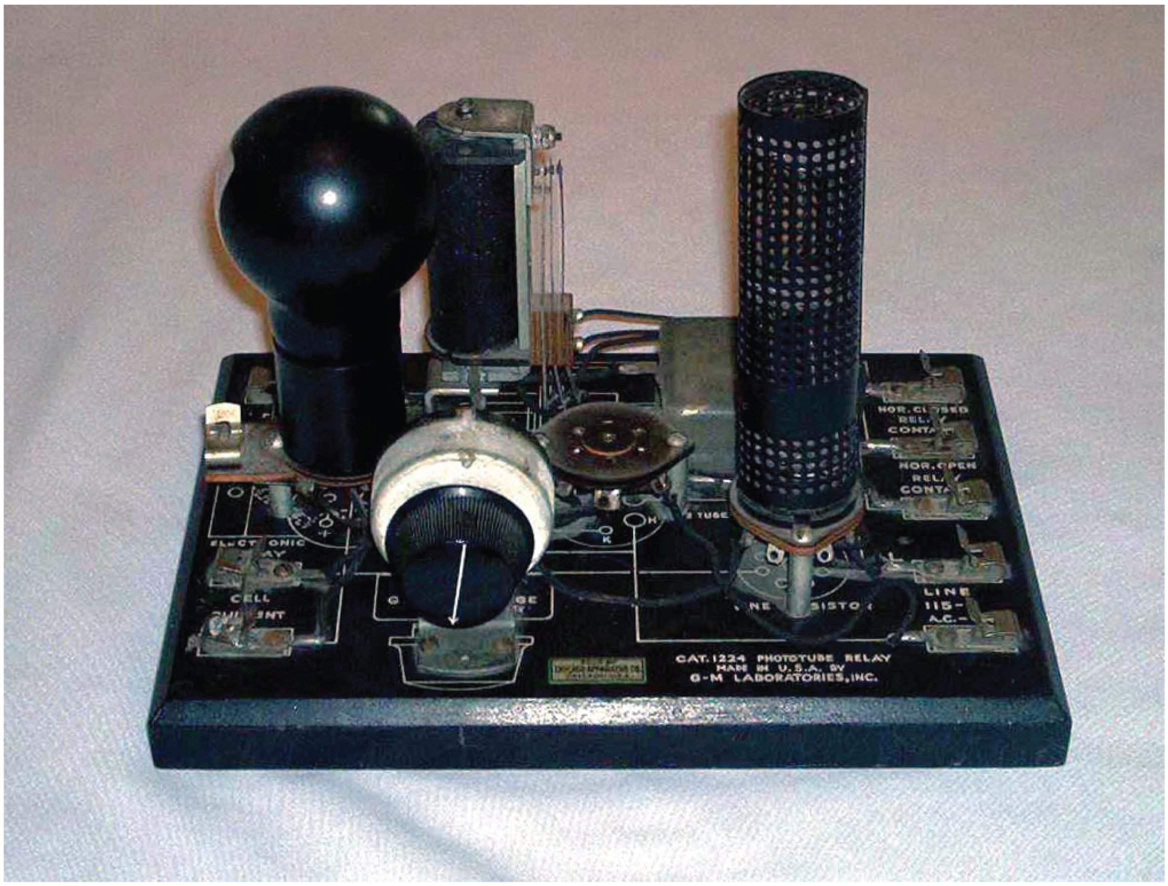
[63]William W. Hackborn, "Projectile motion: Resistance is fertile," Am. Math. Mon. **115**, 813–819 (2008); available at http://www.jstor.org/stable/27642609.

[64]Edward John Routh, *A Treatise on Dynamics of a Particle with Numerous Examples* (Cambridge U.P., Cambridge, 1898), p. 96.

[65]Reference 50, pp. 670–679.

[66]Reference 50, pp. 700–761. Newton reports on his pendulum (pp. 713–723) and vertical fall experiments (pp. 749–761) at great length and with careful attention to details.

[67]Lyle N. Long and Howard Weiss, “The velocity dependence of aerodynamic drag: A primer for mathematicians,” Am. Math. Mon. **106**, 127–135 (1999). For a sphere of diameter d moving at speed s in fluid of density ρ and viscosity μ, the Reynolds number is $Re = s\rho d/\mu$. When $Re \ll 1$, the drag is dominated by viscous forces (yielding Stokes’ Drag Law, $F_D = 6\pi\mu rs$ for a sphere of radius r, with drag proportional to speed; when $Re \gg 1$, inertial forces prevail). Fluid tenacity is expressed by the no-slip condition on the body’s surface (see also Ref. 58).
[68]Neville de Mestre, *The Mathematics of Projectiles in Sport* (Cambridge U.P., Cambridge, 1990), pp. 49–50, 58–60, 123–126, and 135–137.
[69]Reference 3, pp. 311b1–311b5.
[70]Reference 1, pp. xiii–xxxv. Drake firmly adopts this position, calling *Two New Sciences* “a new kind of mathematical physics” (p. xvii) in his introduction to it.
[71]Reference 12, p. 5. Damerow *et al*. assert that the conceptual framework of classical mechanics “begins with such figures as Descartes and Galileo and takes shape with … their successors.”
[72]A. P. French, *Newtonian Mechanics*, The M.I.T. Introductory Physics Series (W.W. Norton & Co., New York, NY, 1971), p. 153. See http://books.wwnorton.com/books/searchresults.aspx?searchtext=newtonian+mechanics.
[73]Reference 4, pp. 31–65. In the physical sciences, Kuhn distinguishes the mathematical tradition from the newer “experimental tradition,” which includes sciences such as chemistry, and claims these traditions “remained distinct” into the 19th century (p. 48). Kuhn fails to consider the motion experiments conducted by Galileo and Newton, which make his claim questionable.
[74]Salomon Bochner, *The Role of Mathematics in the Rise of Science* (Princeton U.P., Princeton, NJ, 1966), p. 7.
[75]Leonhard Euler, “Recherches sur la veritable courbe que décrivent les corps jettés dans l’air ou dans un autre fluide quelconque,” Mem. de L’Acad. Sci. Berlin **9**, 321–352 (1755).
[76]Edward J. McShane, John L. Kelley, and Franklin V. Reno, *Exterior Ballistics* (The University of Denver Press, Denver, 1953), p. 258 and pp. 305–310.
[77]Gilbert Ames Bliss, *Mathematics for Exterior Ballistics* (Wiley, New York, 1944), pp. 17–28.



Phototube Relay

This instrument was made by G-M Laboratories and sold by the Chicago Apparatus Co. The date is in the middle 1930s. The phototube is the spherical object on the left-hand side. A rectangular window has been scratched in the black paint on the outside of the tube, and facing it is a cylindrical cathode of metal, coated with Cesium, which has a low work function. At the center of curvature of the cathode is the anode: a wire at the center of curvature of the cathode. Light striking the cathode produces photoelectrons that are captured by the anode. A [missing] triode vacuum tube amplifies the resulting signal, which then drives the mechanical relay at the left rear. This device is in the Greenslade Collection. (Notes and picture by Thomas B. Greenslade, Jr., Kenyon College)