Oscillations

Restoring Forces

$$\Sigma F = -kx = m\ddot{x} \tag{1}$$

We did see one differential equation in Intro Physics: Hooke's law. And we even had a solution to it.

The general solution for a linear restoring force:

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{2}$$

or, if we use the phase constant: ϕ

$$x(t) = C\cos(\omega t + \phi) \tag{3}$$

[Sim from https://sciencesims.com/sims/plot-adjustable-cos-general]

Detailed Solution of Hooke's Law

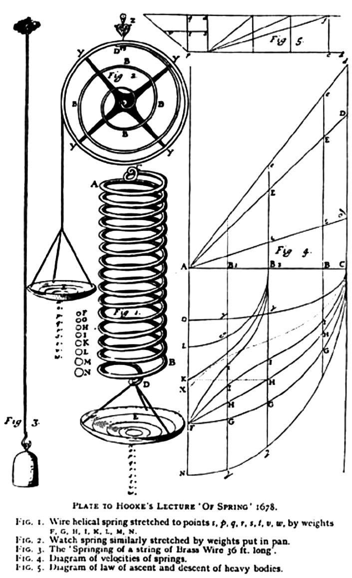


Fig. 1 An illustration from Hooke's On Springs

We'd like to actually solve the diff eq, rather than just 'guessing' at a solution. Here is one method:

Start with the Sums of Forces equation:

$$m\frac{dv}{dt} = -kx \tag{4}$$

This can be modified by noting that

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \tag{5}$$

and so we can write:

$$mv\frac{dv}{dx} = -kx \tag{6}$$

Now we can use the regular separation of variables technique and integrate the results:

$$m\int_{v_0}^{v(x)} v\,dv = -k\int_{x_0}^{x} x dx \tag{7}$$

which will lead to something that should trigger some memories:

$$\frac{1}{2}mv^2 - \frac{1}{2}v_0^2 = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2 \tag{8}$$

Solving for v:

$$v(x) = \sqrt{\frac{k}{m}\left(x_0^2 - x^2 + \frac{v_0^2}{\frac{k}{m}}\right)} \tag{9}$$

or, if we define some of the constants:

$$v(x) = \omega\left(A^2 - x^2\right)^{1/2} \tag{10}$$

with

$$A = \left(x_0^2 + \frac{v_0^2}{\omega^2}\right)^{1/2}$$

and

$$\omega = \sqrt{\frac{k}{m}}$$

We know have $v(x)$, but, we'd really like $v(t)$, so we have to solve:

$$v = \frac{dx}{dt} = \omega\left(A^2 - x^2\right)^{1/2} \tag{11}$$

Separating the variables:

$$\int_{x_0}^{x(t)} \frac{dx}{\left(A^2 - x^2\right)^{1/2}} = \pm\omega \int_0^t dt \tag{12}$$

Performing the integral:

$$\cos^{-1}\left(\frac{x}{A}\right) - \cos^{-1}\left(\frac{x_0}{A}\right) = \mp\omega t$$

Solving for $x(t)$

$$x(t) = A\cos(\omega t + \phi) \tag{13}$$

where

$$\phi = \mp\cos^{-1}\left(\frac{x_0}{A}\right)$$

And since $v = \dot{x}$,

$$v(t) = -\omega A \sin(\omega t + \phi) \tag{14}$$

where

$$\phi = -\tan^{-1}\left(\frac{v_0}{\omega x_0}\right)$$

Equations (13) and (14) have now been shown to be the solutions to Hooke's law.

Another route:

Rewriting Hooke's Law as

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{15}$$

we could attempt to solve it by assuming that the solution must take the form:

$$x(t) = e^{qt}$$

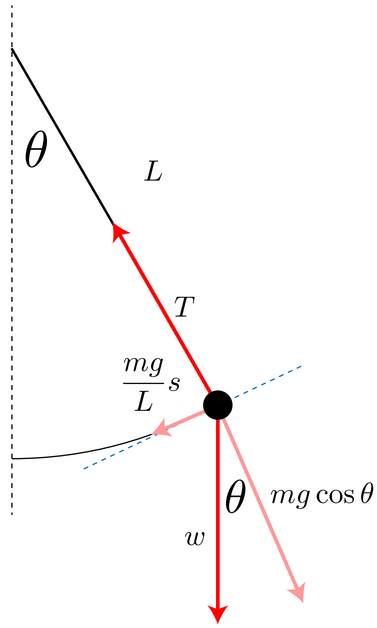
, and then do some work to figure out what q is. (we'll save that...)

Analogous systems

The power of physics comes from its ability to map solutions from one system onto another. Oscillations are a fantastic example of this. The default is always to imagine a spring and a mass, but, there are many other systems that can be treated by the same framework. We just have to get them into a form that looks like $\ddot{x} = -Cx$

Simple Pendulum

For small angles, the tangential force on the pendulum bob will be given by:



The free body diagram of a simple pendulum

$$F_{\text{tang}} = -mg\sin\theta$$

When θ is very small ($\theta < .2$ radians), $\sin\theta \approx \theta$. Using the fact that $s = L\theta$, we can also write:

$$F_{\text{tang}} = -\frac{mg}{L}s$$

This is essentially a restoring force, just like we had for the mass/spring system.

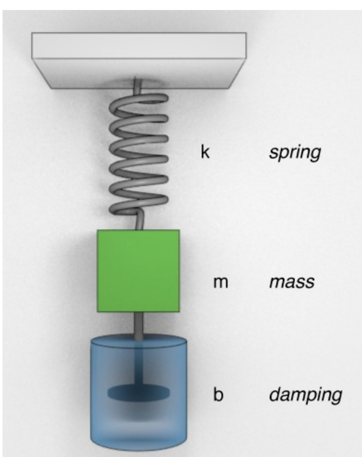
$$F = -kx \quad \Rightarrow \quad F = -\frac{mg}{L}s$$

Other coordinates?

A slightly better coordinate to describe the motion of the pendulum might beθ. That would be a slightly more general coordinate system.

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{16}$$

Damping



Spring-Mass-Damper system

Let's add more terms!

$$F = m\ddot{x} = -kx - b\dot{x}$$

Many real-world scenarios would be better described by a quadratic damping term, $F_{\text{drag}} - cv^2$, but this will make the differential equation*much* harder to solve. So, we'll start with the linear damping term.

Clean it up:

$$\ddot{x} + 2\beta\dot{x} + w_0^2 x = 0 \tag{17}$$

where $\beta \equiv \frac{b}{2m}$ and $\omega_0 \equiv \sqrt{\frac{k}{m}}$

This is a second-order, linear, differential equation. It's solvable if we stipulate two initial conditions:

$$x(0) = x_0$$

and

$$v(0) = v_0$$

A general solution would look like:

$$x \propto e^{\alpha t}$$

If we put this in for x, we would get:

$$\frac{d^2}{dt^2}(e^{\alpha t}) + 2\beta\frac{d}{dt}(e^{\alpha t}) + \omega_0^2 e^{\alpha t} = 0$$

which would, after taking the first and second derivatives of the exponentials:

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0 \tag{18}$$

This is just a quadratic equation in α so we can write the solution to α

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{19}$$

There are 3 possibilities now:

a. Overdamped: $\beta > \omega_0$
b. Critically Damped: $\beta = \omega_0$
c. Underdamped: $\beta < \omega_0$

Each of these possibilities will lead to different solutions for the equation (17),

Overdamped

If

$$\beta > \omega_0$$

then the exponent α in our solution $x = e^{\alpha t}$ will be negative and real.

This means we can write our general solution as the sum of two terms:

$$x(t) = A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t}$$

Where

$$\gamma_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \tag{20}$$
$$\gamma_2 = -\beta - \sqrt{\beta^2 - \omega_0^2} \tag{21}$$

The constants A_1 and A_2 will be determined by the initial conditions of the oscillator, i.e. $x(t=0)$ and $\dot{x}(t=0)$.

Critically Damped

If

$$\beta = \omega_0$$

then solution becomes $x(t) = (A + A't)\,e^{-\beta t}$

This solution comes about because a 2nd order differential equation will need two linearly independent solutions, for the general case.

Underdamped

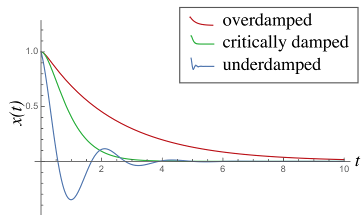
If

$$\beta < \omega_0$$

then we see that $\sqrt{\beta^2 - \omega^2}$ will be *imaginary*.

Let's see how this affects our solution.

One thing to note is that the frequency of oscillation has now changed...



Three cases of damping.

Plotted here are the 3 different regimes of a damped oscillator.

Damped and Driven

Let's add more terms, again!

$$F = m\ddot{x} = -kx - b\dot{x} + F_0 \sin(\omega t)$$

Now we've included a driving force. You can see that it is time dependent. And easy physical system to think of here is when you push a kid on a swing. The swing/kid is just a pendulum, an oscillator. Friction in the chains provides the damping. And you are the driving force, applying a push not all the time, but only at certain times.

Some aspects to note straight away. There are now 3 different frequencies involved in the system:

a. The natural frequency of the SHM: ω_0
b. The damped frequency we just saw in the damping discussion: $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$
c. The driving frequency ω.

Also, this is a new type of differential equation. It's still linear and second order, but now it is *inhomogeneous* because of the because of the driving term on the right (i.e. there is a non-zero constant term like c in this example: $af(x)' + bf(x) + \mathrm{c} = 0$). This will involve a more complicated solution. The basic idea for the solution is that you need to find the sum of two solutions, one general solution for just the *homogeneous* part, and then a particular solution for the *inhomogeneous* part.

In our simple example, the general solution to the homogeneous part would be:

$$af(x)' + bf(x) = 0$$

gives:

$$f(x)_H = Ae^{-\frac{b}{a}x}$$

Then we can look for a particular solution that includes the *inhomogeneous* component. Setting $x = -b/c$ is indeed a solution. The next step would be to define some initial conditions, eg, at $x = 0$, $f(0) = B$. So,

$$f(0) = B = A - \frac{c}{b}$$

Then:

$$f(x) = \left(B + \frac{c}{b}\right) e^{-\frac{b}{a}x} - \frac{c}{b}$$

which cleans up to:

$$f(x) = Be^{-\frac{b}{a}x} + \frac{c}{b}\left(e^{-\frac{b}{a}x} - 1\right)$$

So, now we can solve this:

$$m\ddot{x} + kx + b\dot{x} = F_0 \sin(\omega t)$$

We want our solution to be the sum of the **characteristic** and **particular** solutions:

$$x(t) = x_c(t) + x_p(t)$$

that is, a general solution for the homogeneous part, and a particular solution for the inhomogeneous equation.

We've already found the general solution to the damped oscillator:

$$x_c(t) = Ae^{-\beta t}\cos(\omega_1 t + \phi_0)$$

We'll skip the details of the solution (as they are readily available elsewhere), and simply state the $x_p(t)$ solution as:

$$x_p(t) = C\sin(\omega t + \delta)$$

Together they become:

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t + \phi_0) + C\sin(\omega t - \delta) \tag{22}$$

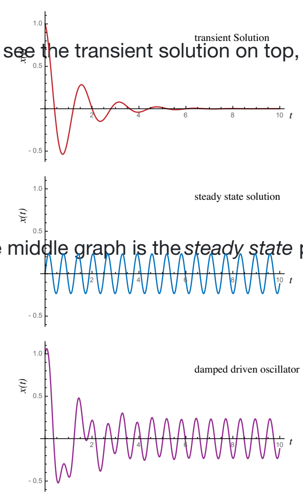
The coefficient C can be found to be:

$$C = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}} \tag{23}$$

This characterizes the amplitude of the steady state solution.

The new parameter in (22) is δ. The physical interpretation of this δ is as follows. We know it functions as a phase constant, meaning it will shift the position of the $\sin$ contribution be some factor. It can be shown to be equal to:

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$



The response of the damed-driven oscillator (bottom) is shown to be the sum of the transient term (top) and the steady state (middle) components.

It's helpful to imagine the combination of the two terms in a graphical form.

We see the transient solution on top, in red. It is an underdamped solution of the damped oscillator. The frequency is given by ω_1 (which is less than the natural resonant frequency ω_0 of the harmonic oscillator:

$$\omega_1 = \sqrt{\omega^2 - \beta^2}$$

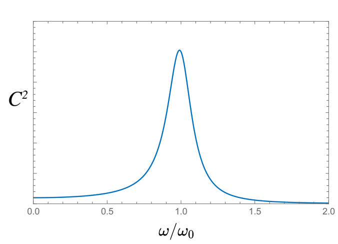
and depends on the damping coefficient.

The middle graph is the *steady state* part of the general solution. This is essentially a sinusoidal function, with an amplitude, frequency, and phase constant.

The sum of the two is shown in the bottom pane, in purple. We can see the combination of the transient and solid state components in the beginning create a irregular pattern, and then after the transient has died out, the forcing function dominates.

Resonance

Here we see the change in the C value as a function of drive frequency (in relation to resonant frequency: $\frac{\omega}{\omega_0}$).



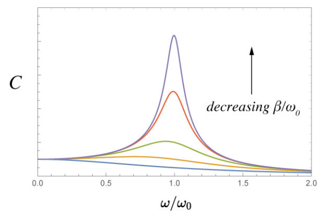
Response of the steady-state component due to changing the drive frequency.

Maximize $C(\omega)$ to find the driving frequency ω that will lead to the largest response.

$$C = \frac{f_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}$$

Taking the derivative of $C(\omega)$ and maximizing should yield:

$$\omega_R = \left(\omega_0^2 - 2\beta^2\right)^{1/2} \tag{24}$$

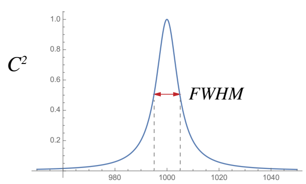


If we decrease the damping (and keep ω_0 constant), we can see the peak of the resonance increasing.

This graph shows 5 different damped driven oscillators. The differences in each curve is the ratio of the damping constant to the resonant frequency. Notice the change in shape.

Quality Factor

Some oscillators are 'better' than others. Of course it's nice to quantify what we mean be better when we do physics. So, we have a parameter called the **Quality Factor**, or Q, of an oscillator.

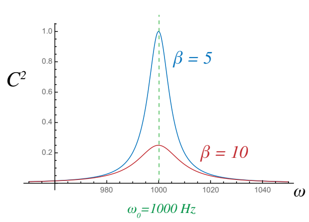


The Full Width Half Max measures describes the shape of the curve.

First we need to articulate the shape of this resonance curve. To do that, we can define the FWHM (Full Width Half Maximum). It's the horizontal distance between the two points that are equal to half the square of the max amplitude.

The actual width of this FWHM can be shown to be approximately equal to the damping coefficient, β.

$$\text{FWHM} \approx 2\beta \tag{25}$$



The Quality Factor is the ratio of the resonant frequency, ω_0, and the damping β.

$$Q = \frac{\omega_0}{2\beta} \tag{26}$$