Numerical Methods - A very brief introduction

Analytic vs. Numeric

$$\frac{dv}{dt} = a = \frac{F}{m}$$

In the case of free fall, this becomes:

$$\dot{v} = g$$

The above equation, $\dot{v} = g$ is technically a differential equation, it just happens to be a very easy one to solve.

$$\frac{dv}{dt} = g$$

Then doing **Separation of Variables**

$$dv = gdt$$

Integrating both sides:

$$\int_{v_O}^{v} dv = \int_0^t g\ dt$$

leads to the familiar equation for velocity of a falling object:

$$v(t) = v_0 + gt$$

This is the analytical solution for velocity using Netwon's second law for the case of a constant acceleration.

The position of the object would can also be solved analytically and you would obtain the familiar 2nd order polynomial:

$$x(t) = x_0 + v_0 t + \frac{1}{2} g t^2$$

More Analytic Solutions:

If $F = -bv$, then we can solve for velocity and obtain:

$$v(t) = v_0 e^{\frac{t}{\tau}}$$

This is another analytic solution, to a slightly more complicated ΣF situation. The harmonic oscillator is another example of an analytic solution:

$$v(t) = A\omega \sin(\omega t + \phi)$$

You've seen a bunch already in fact.

But, what about when

$$\int dv = \int (\text{Something Nasty})\, dt$$

There are many cases in all the branches of physics where the physical system does not lead to a analytically solvable equation. It doesn't mean there's anything wrong with physics, or math. It's perfectly ok. It just means we can't express the answer in a nice way.

Numerical Methods

Re-writing our original definition of acceleration, we can say that:

$$\frac{\Delta v}{\Delta t} = a$$

This is the discrete form, where the differentials are expressed as deltas instead. Expanding out the deltas we would get:

$$v_f - v_i = a(t_f - t_i) = a\Delta t$$

This is essentially how you did physics in HS or at the pre-calc level. But, it can be used to do much more advanced physics too.

Let's say we know a value for v_i

$$v_f - v_i = a\Delta t$$

Then we can just solve this to get a value for v_f at some Δt.

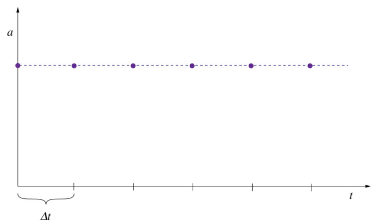
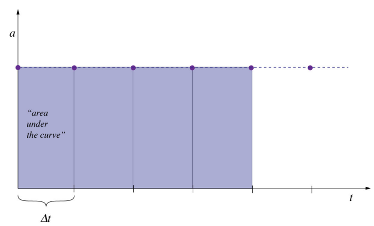


Fig. 1 A plot of a vs. t



The Area under the curve

It will help to consider what we are actually doing when we make that calculation. We are really just finding the area under the $a(t)$ curve. That's what integration is. So the velocity is just the geometric area under the acceleration curve. All we have to do is calculate the area.

Numerical Integration

Let's start with $v_i = 0$. Then we can see that

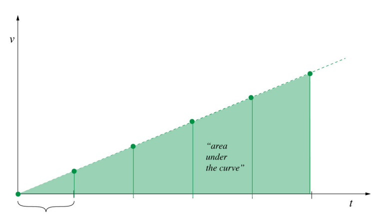
$$v_f = a\Delta t$$

That the area of the first rectangle in the previous graph. Now we can use that v_f in the original equation as the v_i, and we will obtain:

$$v_f = v_i + a\Delta t = a\Delta t + a\Delta t$$

... and so on.

And velocity -> position?



Likewise, we can do the exact same thing for $\frac{dx}{dt} = v$. We calculate the area under the velocity curve to obtain the position at every Δt.

We can see right away though, that our area is a little trickier to calculate here. Now we have triangles!

But, we know the area of a triangle so everything is manageable. We just need to be careful with our choice of velocity.