Euler Lagrange

Euler-Lagrange Equation

$$\frac{\partial F}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'(x)}\right) = 0 \quad (1)$$

$$I = \int_{x_a}^{x_b} F\left[y(x), y'(x), x\right] dx \quad (2)$$

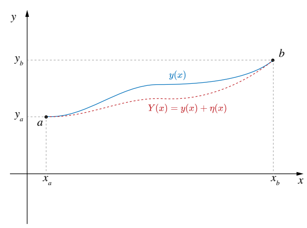

Fig. 1 The right path minimizes the integral. The wrong path does not.

Imagine a *wrong* curve, shown in the dotted blue line.

We can call it

$$Y(x) = y(x) + \eta(x) \quad (3)$$

η is just the difference between the *right* path ($y(x)$) and the *wrong* path.

Since they both start and end at the same two points:

$$\eta(x_a) = \eta(x_b) = 0 \quad (4)$$

Introduce a parameter α:

$$Y(x) = y(x) + \alpha\eta(x) \quad (5)$$

If we set $\alpha = 0$ in (5), then we can see that $Y(x)$ is the *right* path. Thus $I(\alpha)$ is a minimum when $\alpha = 0$.

Now show that

$$\frac{dI}{d\alpha} = 0 \text{ for } \alpha = 0 \quad (6)$$

The integral $I(\alpha)$ becomes, when written out:

$$I(\alpha) = \int_{x_a}^{x_b} F(Y, Y', x) dx \quad (7)$$

or

$$I(\alpha) = \int_{x_a}^{x_b} F(y + \alpha\eta,\ y' + \alpha\eta',\ x)\ dx \quad (8)$$

Now we evaluate $\frac{\partial F}{\partial \alpha}$

$$\frac{\partial F(y + \alpha\eta,\ y' + \alpha\eta',\ x)}{\partial \alpha} \quad (9)$$

The multivariate chain rule:

$$\frac{\partial F}{\partial \alpha} = \frac{\partial F}{\partial Y}\frac{dY}{d\alpha} + \frac{\partial F}{\partial Y'}\frac{dY'}{d\alpha} + \frac{\partial F}{\partial x}\frac{dx}{d\alpha} \quad (10)$$

Since α doesn't depend on x:

$$\frac{dx}{d\alpha} = 0$$

and we will have two terms:

$$\frac{\partial F}{\partial \alpha} = \eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \quad (11)$$

Thus, $dI/d\alpha$:

$$\frac{\partial I}{\partial \alpha} = \int_{x_a}^{x_b} \frac{\partial F}{\partial \alpha} dx = \int_{x_a}^{x_b} \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) dx = 0 \quad (12)$$

This condition is true for any $\eta(x)$ that shares the same start/end points, as in (6).

Integration by Parts

$$\int_{x_a}^{x_b} \eta' \frac{\partial F}{\partial y'} dx = \left[\frac{\partial F}{\partial y'} \eta(x) \right]_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \eta(x) dx \quad (13)$$

Because of (4), the first term on the R.H.S. of (13) is 0, and we have:

$$\int_{x_a}^{x_b} \eta(x) \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) dx = 0 \quad (14)$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \quad (15)$$

Fundamental Lemma of Calculus of Variations:

if

$$\int \eta(x) g(x) dx = 0 \quad (16)$$

then it can be shown that $g(x)$ must be zero.

if not, then assume $g(x)$ is non-zero between x_a and x_b.

If I is an extremum:

$$I = \int_{x_a}^{x_b} F\left[y(x), y'(x), x\right] dx \quad (17)$$

then

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \quad (18)$$

How will we use this?

$$I = \int \left(\frac{1}{2} m v^2 - U \right) dt = \int (T - U)\, dt \quad (19)$$

where

$$T = \frac{1}{2} m v^2 \quad (20)$$

and (for gravity for example)

$$U = mgy \quad (21)$$

The Lagrangian

$$L = T - U \quad (22)$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad (23)$$

leads to:

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x \quad (24)$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = m\ddot{x} \quad (25)$$

so:

$$F_x = m\ddot{x} \quad (26)$$

Do it for all three dimensions and you have

$$-\nabla U = m\mathbf{a} \Rightarrow \mathbf{F} = m\mathbf{a} \quad (27)$$