Coordinates

Cartesian Coordinates

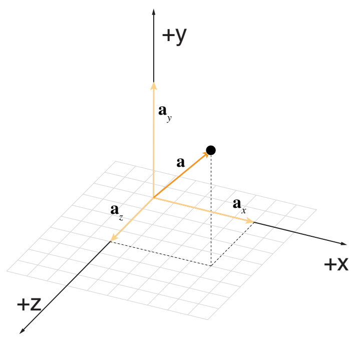


Fig. 1 The Cartesian Coordinate system

The cartesian 3d coordinate system

There are several common ways to express a location:

$$
\begin{aligned}
\mathbf{a} &= a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} && (1)\\
&= a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}} && (2)\\
&= (a_x, a_y, a_z) && (3)\\
&= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 && (4)\\
&= \sum_{i=1}^{3} a_i\mathbf{e_i} && (5)\\
&= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} && (6)
\end{aligned}
$$

In a 3d world, we can describe the position of any point using 3 coordinates.

This of course is a **vector**!

Vector Math

Magnitudes

$a^2 + b^2 = c^2$

Addition & Subtraction

$$\mathbf{a} = a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}$$

And

$$\mathbf{b} = b_1\hat{\mathbf{x}} + b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}}$$

then,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{\mathbf{x}} + (a_2 + b_2)\hat{\mathbf{y}} + (a_3 + b_3)\hat{\mathbf{z}}$$

and

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1)\hat{\mathbf{x}} + (a_2 - b_2)\hat{\mathbf{y}} + (a_3 - b_3)\hat{\mathbf{z}}$$

Dot Product

There are two ways of multiplying vectors:

1. The dot product (or scalar product)

$\mathbf{a} \cdot \mathbf{b} = ab\cos\theta$

$\mathbf{a} \cdot \mathbf{b} = (a_1\hat{\mathbf{x}} + a_2\hat{\mathbf{y}} + a_3\hat{\mathbf{z}}) \cdot (b_1\hat{\mathbf{x}} + b_2\hat{\mathbf{y}} + b_3\hat{\mathbf{z}})$

$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Dot Product - Linear Algebra

$$[\,a_1\ a_2\ a_3\,]\cdot\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=a_1b_1+a_2b_2+a_3b_3$$

Cross Product

Defined as:

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)\ \mathbf{n}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{n}$ is the unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$

This produces a third vector that points perpendicular to both the original vectors.

$$\mathbf{a}\times\mathbf{b}=(a_x\hat{\mathbf{x}}+a_y\hat{\mathbf{y}}+a_z\hat{\mathbf{z}})\times(b_x\hat{\mathbf{x}}+b_y\hat{\mathbf{y}}+b_z\hat{\mathbf{z}})$$

We can calculate the cross product by computing the determinant of this matrix:

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{x}}&\hat{\mathbf{y}}&\hat{\mathbf{z}}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}$$

$$\mathbf{a}\times\mathbf{b}=(a_yb_z-a_zb_y)\hat{\mathbf{x}}+(a_zb_x-a_xb_z)\hat{\mathbf{y}}+(a_xb_y-a_yb_x)\hat{\mathbf{z}}$$

Mathematica:

```
a = {a1, a2, a3}
b = {b1, b2, b3}
Dot[a,b]
-> a1 b1 + a2 b2 + a3 b3
Cross[a,b]
-> {-a3 b2 + a2 b3, a3 b1 - a1 b3, -a2 b1 + a1 b2}
```

numpy (python)

```
import numpy as np

a = np.array([2,4,5])
b = np.array([1,-2,3])
c = np.dot(a,b)
print(c)
-> 9
d = np.cross(a,b)
print(d)
-> [22 -1 -8]
```

Quadratic Drag in two dimensions

In cartesian coordinates:

$$\begin{aligned}m\ddot{\mathbf{r}}&=m\mathbf{g}-cv^2\ \hat{\mathbf{v}} &(7)\\&=m\mathbf{g}-cv\ \mathbf{v} &(8)\end{aligned}$$

This, if you remember 2 kinematics from 20700, requires breaking the problem up into both directions. Previously, those two directions where independent, meaning what happens in one, stays there and doesn't affect the other. But, know, with the velocity dependent drag, they are coupled:

$$m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\, v_x \quad (9)$$
$$m\dot{v}_y = -mg - c\sqrt{v_x^2 + v_y^2}\, v_y \quad (10)$$

This is an example (already!) of something that has no analytic solution. That means we can't use separation of variable or any other regular diff.eq techniques to find a general solution to this. Numerical methods are the only way.

Example Problem #1:

Plot motion for a projectile experiencing quadratic drag.

Mathematica:

Define some parameters

```
r = .05;
area = Pi r^2;
m = .15;
g = 9.8;
Cd = 0.47;
\[Rho] = 1.29;
c = 1/2 \[Rho] area Cd;
\[Theta] = Pi/4
v0 = 40;
```

Define our differential equations

In the x direction:

$$m\ddot{x} = -c\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}$$

And

$$m\ddot{y} = -mg - c\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}$$

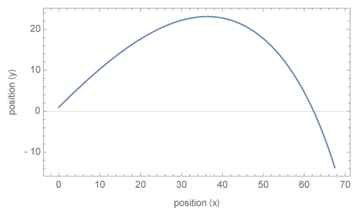
```
diffeqs = {m x''[t] == -c x'[t]*Sqrt[x'[t]^2 + y'[t]^2],
  m y''[t] == -m g - c y'[t]*Sqrt[x'[t]^2 + y'[t]^2],
  x[0] == 0,
  x'[0] == v0 Cos[\[Theta]],
  y[0] == 1,
  y'[0] == v0 Sin[\[Theta]]
  }
```

Use `NDSolve[]` to solve them.

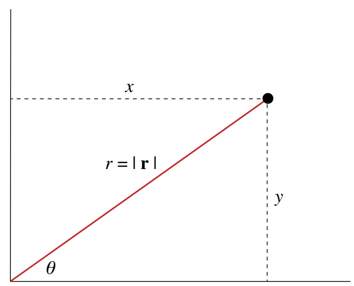
```
solutions = NDSolve[diffeqs, {x[t], y[t]}, {t, 0, 6}]
```

Plot x vs. y using `ParametricPlot[]`

```
ParametricPlot[{x[t] /. solutions[[1, 1]], y[t] /. solutions[[1, 2]]}, {t, 0, 5}]
```



Other Coordinate systems



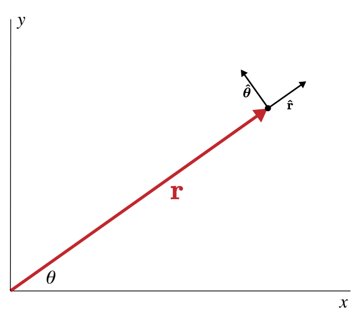
The basic polar coordinate system

$$x = r\cos\theta \quad (11)$$
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \quad (12)$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Goal: Express newton's laws in these non-cartesian coordinates:

$$F = m\ddot{\mathbf{r}}$$

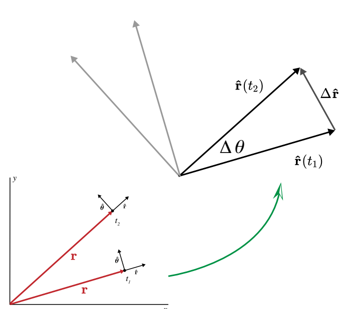


Unit vectors in polar coordinates

Unit vectors in polar coordinates

$\hat{\mathbf{r}}$ points in the direction of increasing r (given a fixed θ)

$\hat{\boldsymbol{\theta}}$ points in the direction of increasing θ (given a fixed r)



Now, the unit vectors change!

We start with $\hat{\mathbf{r}}$

$$\Delta\hat{\mathbf{r}} \approx \Delta\theta\,\hat{\boldsymbol{\theta}} \quad (13)$$
$$\approx \dot{\theta}\Delta t\,\hat{\boldsymbol{\theta}} \quad (14)$$

Taking the time derivative:

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\hat{\boldsymbol{\theta}}$$

Taking the time derivative of $\mathbf{r} = r\hat{\mathbf{r}}$

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \quad (15)$$

which can be used to express the velocity of the particle in polar coordinates:

$$\mathbf{v} \equiv \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\ \hat{\boldsymbol{\theta}} \quad (16)$$

This might look a little different, but you've seen it before. The components are:

$$v_r = \dot{r} \quad (17)$$
$$v_\theta = r\dot{\theta} = r\omega \quad (18)$$

(18) is just what we had for **Uniform Circular Motion**, i.e. when r wasn't changing. Now we have to account for changing r values, so we really need both terms.

Velocity expressed in polar coordinates:

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\ \hat{\boldsymbol{\theta}} \quad (19)$$

Now we need to get the acceleration

To find the acceleration, we know have to take the time derivative of the velocity:

$$\mathbf{a} \equiv \ddot{\mathbf{r}} = \frac{d}{dt}\dot{\mathbf{r}} = \frac{d}{dt}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\ \hat{\boldsymbol{\theta}}\right) \quad (20)$$

The new term to deal with is

$$\frac{d\hat{\boldsymbol{\theta}}}{dt}$$

which by similar arguments as above, we can see will be:

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}} \quad (21)$$

Add this back into our acceleration equation:

$$\mathbf{a} = \left(\ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt}\right) + \left(\left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}} + r\dot{\theta}\frac{d\hat{\boldsymbol{\theta}}}{dt}\right) \quad (22)$$

Acceleration in polar coordinates:

$$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}} \quad (23)$$