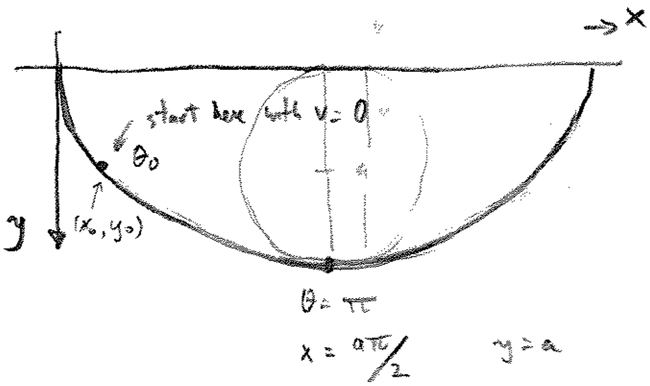
2) $x = \frac{a}{2}(\theta - \sin\theta)$

$y = \frac{a}{2}(1 - \cos\theta)$



find E of particle

$$E = \tfrac{1}{2}mv^2 + mg(-y) = -mgy_0$$

$$\therefore \quad v = (2g(y - y_0))^{1/2}$$

$$t = \int \frac{ds}{v} \quad \leftarrow \quad \left(v = \frac{ds}{dt} \right)$$

$$t = \int_{\theta=\theta_0}^{\pi} \frac{(dx^2 + dy^2)^{1/2}}{(2g(y - y_0))^{1/2}} \quad \text{, now}$$

find $\frac{dx}{d\theta}$ of $x = \frac{a}{2}(\theta - \sin\theta)$

$\frac{dy}{d\theta}$ of $y = \frac{a}{2}(1 - \cos\theta)$

$$dx = \frac{a}{2}(1 - \cos\theta)\,d\theta \;;\quad dy = \frac{a}{2}(\sin\theta)\,d\theta$$

which leads to $dx^2 + dy^2 = \frac{a^2}{2}(1 - \cos\theta)\,d\theta^2$

and $y - y_0 = \frac{a}{2}(\cos\theta - \cos\theta_0)$

a) $$t = \int_{\theta=\theta_0}^{\pi} \frac{\sqrt{\frac{a^2}{2}(1-\cos\theta)\,d\theta^2}}{\sqrt{2g\frac{a}{2}(\cos\theta - \cos\theta_0)}} = \sqrt{\frac{a}{2g}} \int \sqrt{\frac{1-\cos\theta}{\cos\theta - \cos\theta_0}}\, d\theta$$

b) now, $1 - \cos\theta = 2\sin^2\frac{\theta}{2} = 2\sin^2\varphi$ if $\varphi = \theta/2$ and $\frac{d\theta}{d\varphi} = 2 \quad \therefore d\theta = 2\,d\varphi$

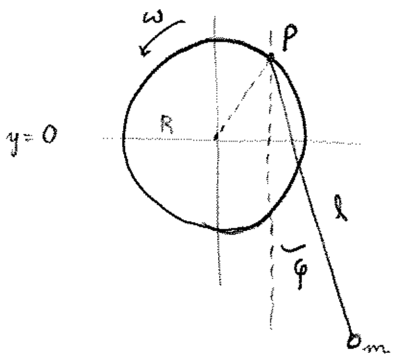
$$t = \sqrt{\frac{a}{2g}} \int_{\varphi_0}^{\pi/2} \frac{2\sin\varphi}{\sqrt{\cos^2\varphi_0 - \cos^2\varphi}}\, d\varphi$$

now, if $\cos\varphi = z \quad \therefore \frac{dz}{d\varphi} = -\sin\varphi$

sub let $\frac{z}{z_0} = v$ $\therefore dz = z_0\,dv$

$$t = 2\sqrt{\frac{a}{2g}} \int_0^{z_0} \frac{dz}{(z_0^2 - z^2)^{1/2}} = 2\sqrt{\frac{a}{2g}} \int_0^1 \frac{dv}{(1 - v^2)} = \pi\sqrt{\frac{a}{2g}}$$

3



first, find the position of the bob in x & y

$$r_x = R\cos\omega t + l\sin\varphi$$

$$r_y = R\sin\omega t - l\cos\varphi$$

therefore the velocity is

$$V_x = \frac{dr_x}{dt} = -\omega R\sin\omega t + \dot\varphi l\cos\varphi$$

$$V_y = \frac{dr_y}{dt} = \omega R\cos\omega t + \dot\varphi l\sin\varphi$$

a) thus the Lagrangian is

$$L = \frac{1}{2}mv^2 - mgy = \frac{1}{2}m\left[(-\omega R\sin\omega t + \dot\varphi l\cos\varphi)^2 + (\omega R\cos\omega t + \dot\varphi l\sin\varphi)^2\right]$$
$$- mg(R\sin\omega t - l\cos\varphi)$$

using $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ and $\sin^2\theta + \cos^2\theta = 1$

$$= \frac{1}{2}m\left[\omega^2R^2 + \dot\varphi^2l^2 + 2\omega R\dot\varphi l\sin(\varphi-\omega t)\right] - mg(R\sin\omega t - l\cos\varphi)$$

b) since φ is the only variable: $\frac{\partial L}{\partial\varphi} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) = 0$

$$\frac{\partial L}{\partial\varphi} = m\omega R\dot\varphi l\cos(\varphi-\omega t) - mgl\sin\varphi$$

$$\frac{\partial L}{\partial\dot\varphi} = m\left[\dot\varphi l^2 + \omega Rl\sin(\varphi-\omega t)\right] \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) = m\left[\ddot\varphi l^2 + \omega Rl\cos(\varphi-\omega t)(\dot\varphi-\omega)\right]$$

$$\therefore\ \underbrace{m\omega R\dot\varphi l\cos(\varphi-\omega t) - mgl\sin\varphi = m[\ddot\varphi l^2 + \omega Rl\cos(\varphi-\omega t)(\dot\varphi}_{\text{goes away}} - \omega)]$$

$$-mgl\sin\varphi = \ddot\varphi l^2 - \omega^2Rl\cos(\varphi-\omega t)$$

$$\therefore\ \ddot\varphi l = -g\sin\varphi + \omega^2R\cos(\varphi-\omega t)$$

if $\omega = 0 \rightarrow \ddot\varphi = -\frac{g}{l}\sin\varphi$, a.k.a. the simple pendulum.

4) Atwood machine w/ massive pulley:

$$I_{pulley} = \frac{1}{2}MR^2 \rightarrow KE = \frac{1}{2}I\omega^2 = \frac{1}{2}I\dot{x}^2/R^2 \quad \text{since } \omega = \frac{\dot{x}}{R}$$

We must include the rotational kinetic energy of the pulley in $\mathcal{L}$.

$$\mathcal{L} = T - V = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 - \left(-(m_1 - m_2)gx\right)$$

$$= \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2)gx$$

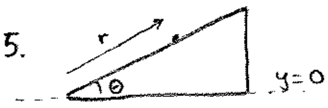
$$\frac{\partial L}{\partial x} = (m_1 - m_2)g$$

$$\frac{\partial L}{\partial \dot{x}} = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x}$$

$$\therefore \ddot{x} = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)} = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{M}{2}\right)}$$

i.e. → slowly than ideal atwood w/ no mass in pulley

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2}g \quad (\text{massLESS pulley})$$

5. 

first, find the velocity of the mass in polar coordinates:

$$v^2(r,\theta) = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\mathcal{U} = mgr\sin\theta$$

thus, our Lagrangian is $L = T - \mathcal{U}$

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\sin\theta$$

$$\text{let } \underbrace{\dot{\theta} = \omega}_{\text{constant}} \implies \theta = \omega t$$

thus

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + \omega^2 r^2) - mgr\sin(\omega t)$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \implies \frac{\partial L}{\partial r} = \tfrac{1}{2}m\omega^2 2r - mg\sin(\omega t)$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\therefore \quad m\ddot{r} = m\omega^2 r - mg\sin(\omega t)$$

or $\ddot{r} - \omega^2 r = -g\sin(\omega t)$ } Non-homogeneous Diff Eq

General solution $\Rightarrow r = r_H + r_P$ (Homogeneous & Particular)

(Homogeneous) $\longrightarrow \ddot{r} - \omega^2 r = 0 \implies r_H = Ae^{\omega t} + Be^{-\omega t}$

(Particular) $\longrightarrow$ try $r_p = C\sin\omega t$, then $\ddot{r}_p = -C\omega^2\sin(\omega t)$

$\therefore \quad -C\omega^2\sin(\omega t) - \omega^2 C\sin\omega t = -g\sin(\omega t)$ solve for C

$$C = \frac{g}{2\omega^2}, \text{ then } r(t) = Ae^{\omega t} + Be^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$$

Now, initial conditions $\Rightarrow r(0) = r_0$ & $\dot{r}(0) = 0$

thus $r_0 = A + B$ & $0 = A - B + \frac{g}{2\omega^2} \implies$ Solve for A & B

$$A = \tfrac{1}{2}\left[r_0 - \frac{g}{2\omega^2}\right] \quad \& \quad B = \tfrac{1}{2}\left[r_0 + \frac{g}{2\omega^2}\right]$$

leads to: $r = \tfrac{1}{2}\left[r_0 - \frac{g}{2\omega^2}\right]e^{\omega t} + \tfrac{1}{2}\left[r_0 + \frac{g}{2\omega^2}\right]e^{-\omega t} + \frac{g}{2\omega^2}\sin(\omega t)$

or: $r(t) = r_0\cosh(\omega t) + \frac{g}{2\omega^2}\left(\sin(\omega t) - \sinh(\omega t)\right)$