#1. Prove $e^{i\theta} = \cos\theta + i\sin\theta$ using Taylor Series.

for $e^{i\theta}$: Taylor is $e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

where $f^{(n)}(a)$ is the n^{th} derivative at a

thus, for $a=0$, n^{th} derivative is $=1$

$$e^x = \sum_{n=0}^{\infty} \frac{(1)(x)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

for $\cos\theta$, one obtains

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

and for $\sin\theta$

$$\sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$$

now use $e^{i\theta}$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \dots$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right)}_{=\cos\theta} + \underbrace{i}_{i} \underbrace{\left(\theta - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}_{\sin\theta}$$

$$= \cos\theta \quad + \quad i \quad \sin\theta$$

$\therefore$ $e^{i\theta} = \cos\theta + i\sin\theta$

#2. from eq (8) of the oscillator notes:

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$

this is essentially a statement of cons. of mech. energy.

$$\Delta KE = \Delta ME \quad \text{or}$$

$$\text{total } E_0 = \text{total } E_1$$

$\therefore \quad \frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$ (replacing $x \to x_2$ & $x_1 \to x_1$)

thus $\quad m(v_1^2 - v_2^2) = k(x_2^2 - x_1^2)$ (eq. 1)

v_1, x_1 are the vel and position at time 1

v_2, x_2 are the vel and position at time 2

thus since $\omega^2 = \frac{k}{m}$, eq (1) shows

$$\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} \quad \checkmark$$

and since the total energy is equal to $E = \frac{1}{2}kA^2$

(because at $x = A$, $v = 0$)

then

$$A^2 = \frac{m}{k}v_1^2 + x_1^2 = \left(\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}\right)v_1^2 + x_1^2$$

$$= \frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}$$

$$A = \left(\frac{x_2^2 v_1^2 - x_1^2 v_2^2}{v_1^2 - v_2^2}\right)^{1/2} \quad \checkmark$$

#3 Solve the simple harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad (1.1)$$

a) let $x = e^{qt}$ (1.2)

$$\frac{d^2x}{dt^2} = q^2 e^{qt}$$

thus $q^2 e^{qt} = -\omega^2 e^{qt}$

or $q^2 = -\omega^2$

or $q = \pm\sqrt{-\omega^2} = \pm i\omega$ ✓

b) thus $x(t) = a_1 e^{i\omega t} + a_2 e^{-i\omega t}$ (1.3)

if $x(0) = x_0$ & $v(0) = v_0$ $\left(v = \frac{dx}{dt}\right)$

$x_0 = a_1 + a_2$ and $v_0 = a_1 i\omega - a_2 i\omega$ (evaluate 1.3 at initial cond.)

$\downarrow$

solve for a_2

$$a_2 = a_1 - \frac{v_0}{i\omega} = a_1 + \frac{iv_0}{\omega}$$

$\therefore$ $x_0 = a_1 + a_1 + \frac{iv_0}{\omega}$

$\downarrow$

$a_1 = \frac{1}{2}\left(x_0 - \frac{iv_0}{\omega}\right)$, likewise $a_2 = \frac{1}{2}\left(x_0 + \frac{iv_0}{\omega}\right)$