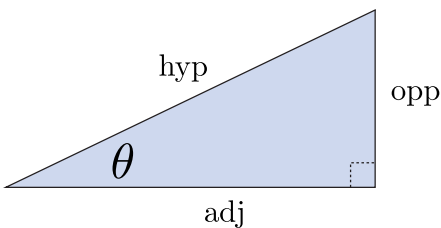# Math, Trigonometry and Vectors

## Geometry

### Trigonometry
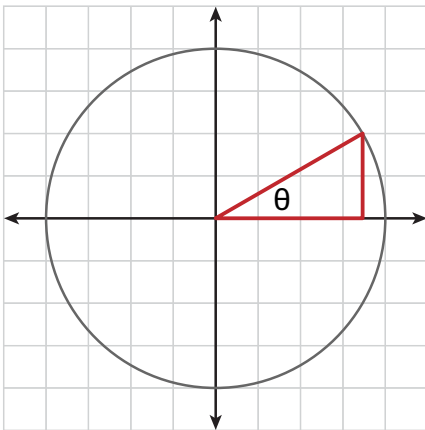

This picture represents the origins of the trigonometric relations. Inscribe a right triangle inside a circle. Now, the angle theta is one of the angles of the triangle. The ratios of the various sides make up the trig functions.

### Trig Definitions


Here's a familiar image.

To make predictive models of the physical world, we'll need to make visual models, which we can then turn into analytical statements. Many of the models will be geometric in nature. Thus, we'll need things like the trigonometric relations to establish relations between the components. In the triangle shown here, one angle is marked with a $\theta$. The sides are labeled in relation to this angle: opposite, adjacent, and hypotenuse. So, $\sin(\theta)$ is defined as the ratio of the side labeled *opposite* to the side known as the *hypotenuse*.

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}}$$

.
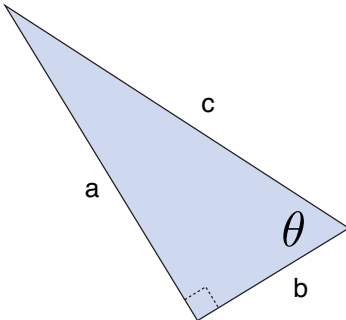
$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}}$$

.

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

.
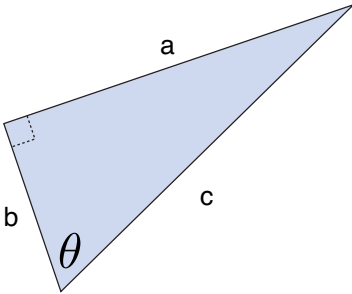
This one might be a little less familiar, but the same rules apply.

## Inverse functions

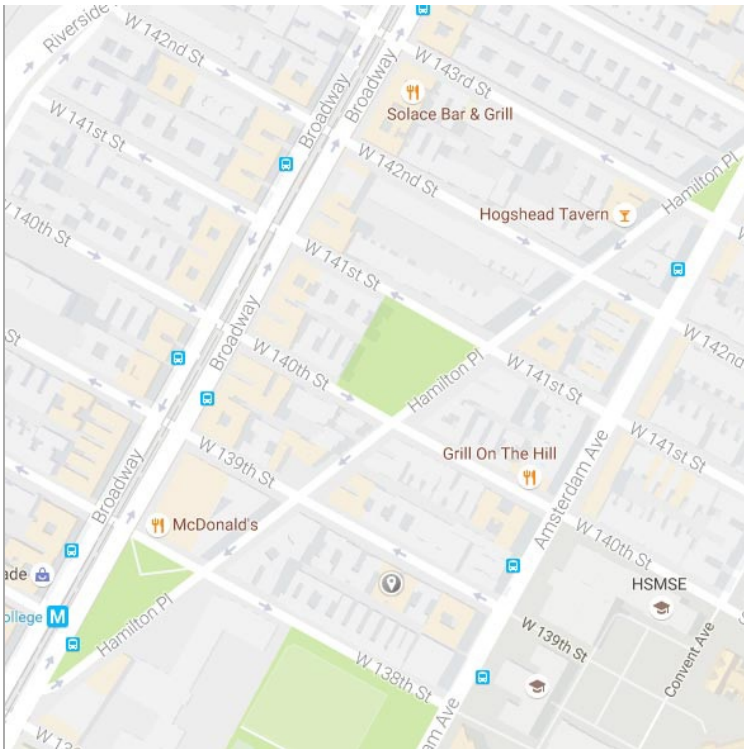We can also use the inverse trigonmetric functions.

The inverse trig functions take the ratio of lengths (a dimensionless number) and return an angle (in degrees or radians).

$$\sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right) = \theta$$

$$\cos^{-1}\left(\frac{\text{adj}}{\text{hyp}}\right) = \theta$$

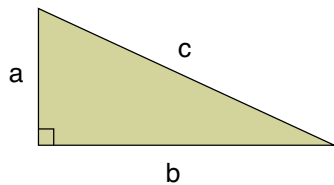$$\tan^{-1}\left(\frac{\text{opp}}{\text{adj}}\right) = \theta$$

Example Problem #1:

It's 450 meters from the corner of Hamilton Place and Broadway to 142nd St and Broadway. It's 489 meters from the same corner to 142nd if you walk along Hamilton Place. a) What is the angle between Hamilton and Broadway?

$$\theta = \cos^{-1}\left(\frac{450}{489}\right) = 23.04°$$
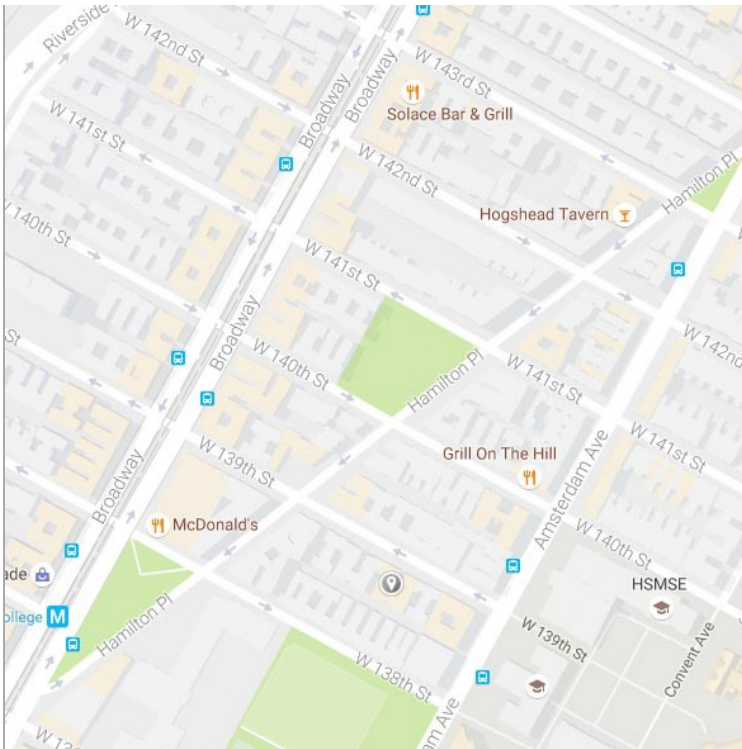
## Pythagorean Theorem



We'll use this relationship *all the time*.

$$a^2 + b^2 = c^2$$

. This is also known as Euclid's 47th proposition from the first book of the Elements.

Example Problem #2:

It's 450 meters from the corner of Hamilton Place and Broadway to 142nd St and Broadway. It's 489 meters from the same corner to 142nd if you walk along Hamilton Place. b) How far is it from Broadway to Hamilton Pl. walking along 142nd St?

$$b = \sqrt{c^2 - a^2} = \sqrt{489^2 - 450^2} = 191.4 \text{ m}$$

map

# Scalars and Vectors

These are two different mathematical or physical entities.

**Scalars:** A scalar quantity is completely specified by a single value with an appropriate unit and has no direction. (e.g. $20)

**Vectors:** A vector quantity is completely described by a number and appropriate units plus a direction. (e.g. person walks 2 km E)

## Some Examples:

Scalars: Temperature, Speed, Distance, length, density

Vectors: Displacement, Velocity, Force, Weight

When thinking out the physical world, you should intuitively notice that certain quantities or phenomena have different effects depending on which way they are pointed, in other words, their effects depend on their direction. For example, it's a lot easier to walk with the wind blowing in the same direction as your motion, rather than the other way: walking against the wind. There are two vector quantities at play in this example. Your direction of motion (that would be inferred from your velocity vector) and the velocity of the wind. When they point in the same direction, your motion is aided by the wind, when they are in opposite directions, your motion is impeded. Not only do the magnitudes of these two quantities matter, but so do their directions. And thus, we need to use vectors.

## Vector vs. Scalar example

A particle travels from A to B along the path shown by the dotted red line. This is the **distance traveled** and is a *scalar*

The **displacement** (change in position) is the solid line from (a) to (b). The displacement is independent of the path taken between the two points displacement is a *vector* (it has length and direction).
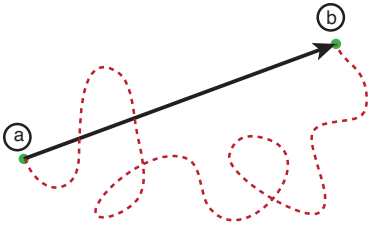
This image illustrates the difference between displacement and distance traveled. Looking at the dotted line, which represents the distance traveled, compared to the solid displacement vector, we can see that a) its magnitude is probably much larger than the displacement vector, and b) it doesn't have a clear direction associated with it.

### Notation

When writing math by hand, just put an arrow on top of the variable. This will indicate it is a vector: $\vec{A}$

In printed text, you'll see vectors either in bold face: $\mathbf{A}$ or with an arrow: $\vec{A}$

If we want to refer to the magnitude only of a vector quantity, we can use absolute value bars: $|A|$, or just in italics: $A$.
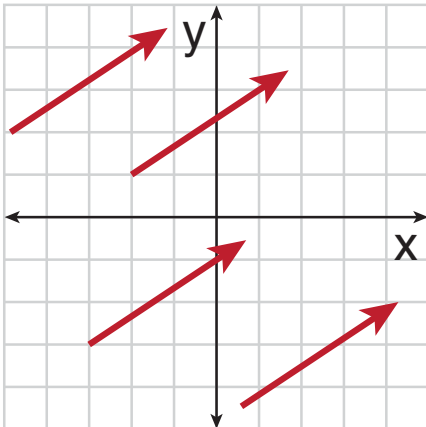
Wind Map

# Properties of vectors

Two vectors are equal if they have the same magnitude and the same direction

$\mathbf{A} = \mathbf{B}$ if $|A| = |B|$ and they point along parallel lines

All of the vectors shown are equal in magnitude and direction, thus they are *equal.*



It might be helpful to think of vectors a notation like this: $\mathbf{A} = (\mathrm{mag}, \mathrm{dir})$. Thinking this way we can see that the definition of the vector only requires two elements, the magnitude and the direction. If we can another vector $\mathbf{B}$, with the same magnitude and direction, it would necessarily have to be equal to $\mathbf{A}$.

## Addition of Vectors

Adding two scalar quantities is easy. We just add them like we would add any normal quantity.

However, vectors involve more math. We have to also take into account which way they are pointing.

Example Problem #3:
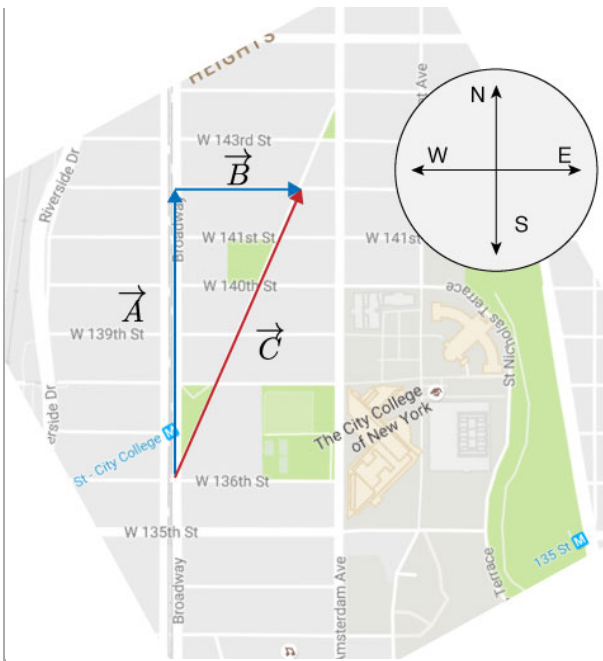
Imagine we walk along two displacement vectors $\vec{A}$ and $\vec{B}$. What is the *resultant* displacement? Or, what is $\vec{A} + \vec{B}$?

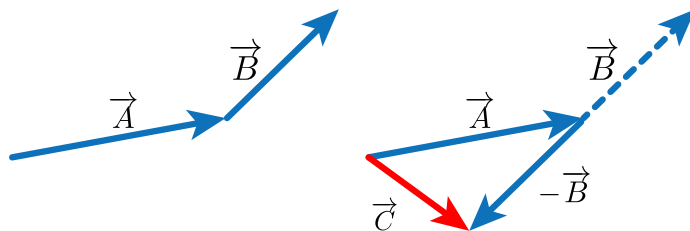## Vector Addition: Graphically

To add $\vec{A} + \vec{B}$

1. Arrange the vectors tip to tail.
2. Connect the tip of $\vec{A}$ to the origin of $\vec{B}$.
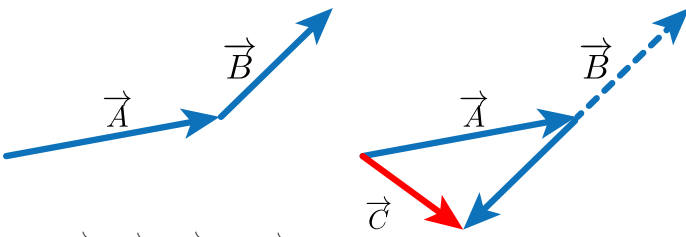
Link to Vector Addition Sim.

## Negative of a Vector

The negative of a vector is simply a vector with the same magnitude, but pointed in the opposite direction.



The resultant of $\vec{A} + (-\vec{A}) = 0$
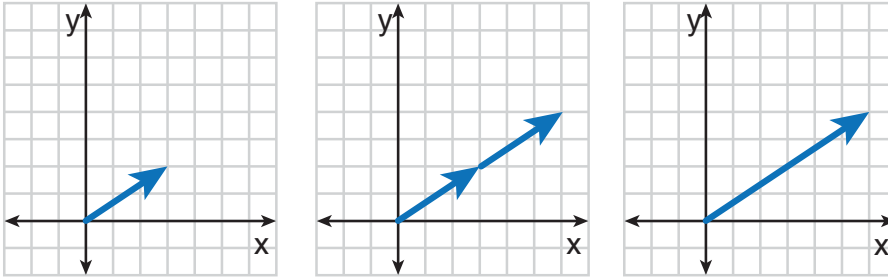
## Vector Subtraction

To Subtract two vectors, say $\vec{A} - \vec{B}$, all we need to do is add the negative of $\vec{B}$ to $\vec{A}$.

Since, $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

## Multiplication of a Vector times a Scalar

The result of the multiplication or division by a scalar is a vector.

$$\mathbf{A} \times 2 = \mathbf{A} + \mathbf{A}$$

The magnitude of the vector is multiplied or divided by the scalar.
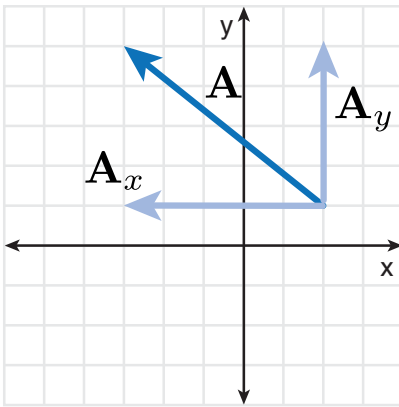
## Vector Components

The components of a vector are the parts of a vector that point along a given axis. We'll use the Cartesian Coordinate System most often.

Here we see the $x$ and $y$ components of the vector $\mathrm{A}$.

We can see that the $x$ component, $\mathrm{A}_x$, points all along the $x$ axis, while the $y$ component, $\mathrm{A}_y$, points only along the $y$ axis.
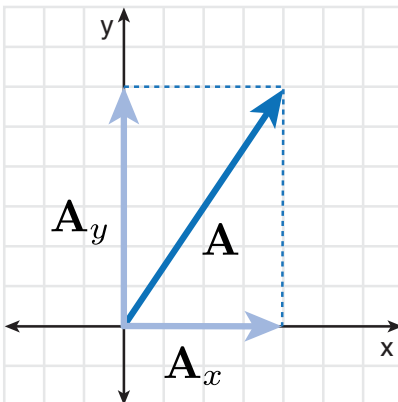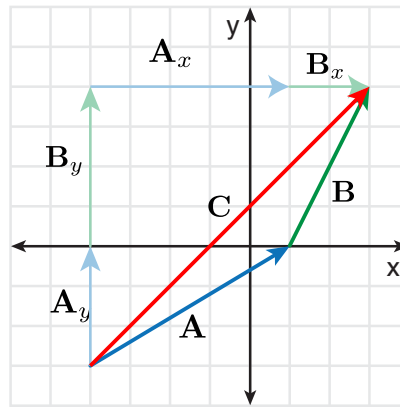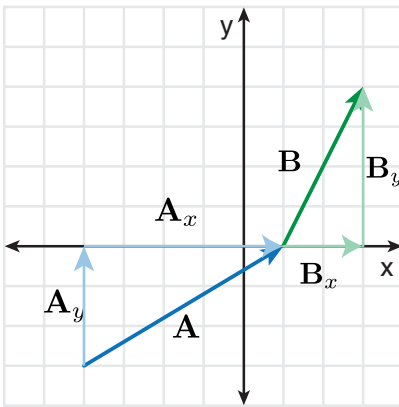
$$A_y = +4$$
$$A_x = -5$$

Here's another vector $A$ *decomposed* into its $x$ and $y$ components.

We can use these vector components to add two arbitrary vectors together. (notice that $A$ and $B$ are not at right angles to each other.)



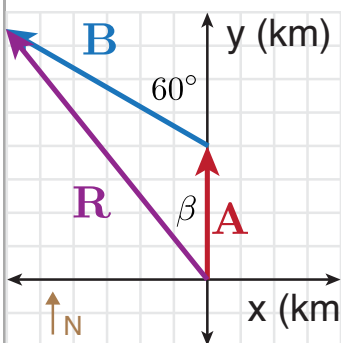We'll combine the components of $A$ and $B$ to get the components of $C$.

$$C_x = A_x + B_x$$

$$C_y = A_y + B_y$$

Once we have the components of $C$, we can use the pythagorean theorem to get the magnitude of $C$.
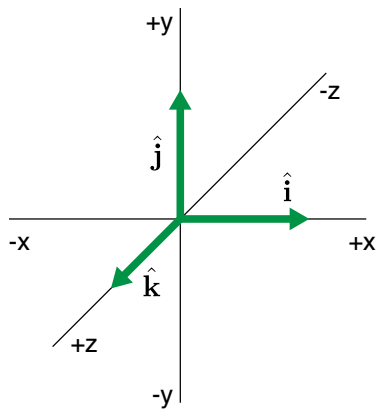
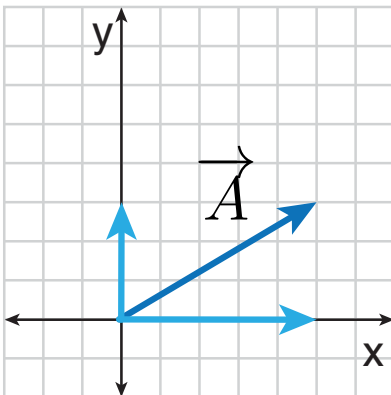$$|C| = \sqrt{C_x^2 + C_y^2}$$

**Example Problem #4:**



A car travels 20 km due N and then 35 km in a direction 60° W of N. Find the magnitude and direction of the car's resultant displacement.

# Unit vectors

A **unit vector** is a vector that has a magnitude of exactly 1, and points in a given direction.

## Function of Unit vectors



Rather than always using the $\theta$ and magnitude of a vector to describe it, we can use the unit vectors.

$\mathrm{A} = A_x\hat{\mathrm{i}} + A_y\hat{\mathrm{j}}$

$\mathrm{A} = 5\hat{\mathrm{i}} + 3\hat{\mathrm{j}}$

## Dot Product

There are two ways of multiplying vectors:

1. The dot product (or scalar product)

$\mathrm{a} \cdot \mathrm{b} = ab\cos\theta$

$\mathrm{a} \cdot \mathrm{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}})$

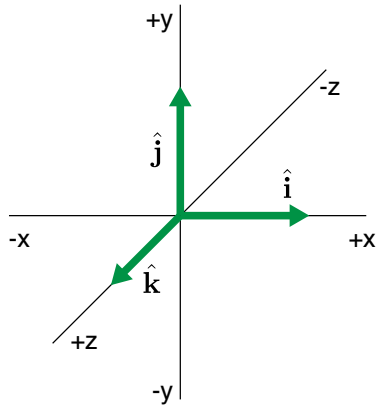$\mathrm{a} \cdot \mathrm{b} = a_xb_x + a_yb_y + a_zb_z$
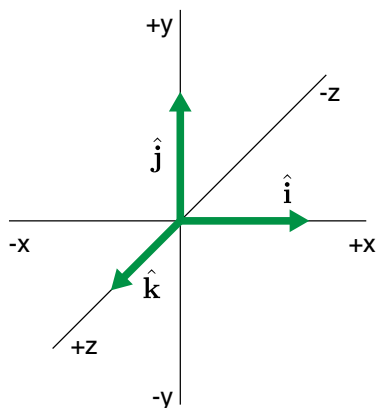
## Cross Product

The second method produces another vector:

2. The cross product (or vector product)

$$|\mathbf{a} \times \mathbf{b}| = ab\sin\phi$$

This produces a third vector that points perpendicular to both the original vectors.

$$\mathbf{a} \times \mathbf{b} = (a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}) \times (b_x\hat{\mathbf{i}} + b_y\hat{\mathbf{j}} + b_z\hat{\mathbf{k}})$$
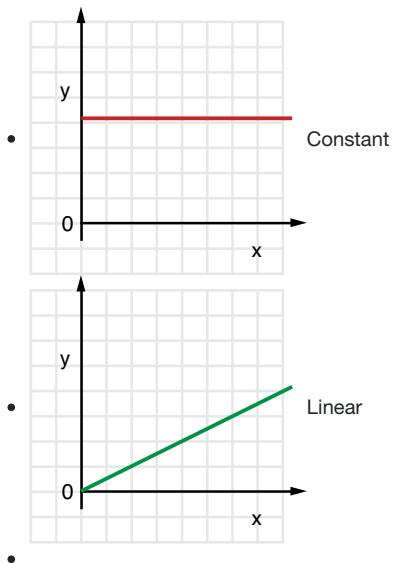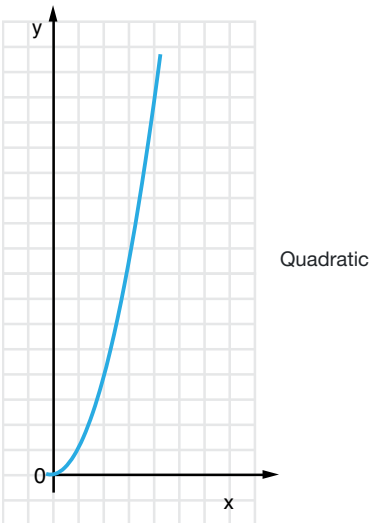
# Other math

## Graphs & Plots

Visual Information is **Powerful**.

## Functional Relations

Quadratic

## Simultaneous Solutions

Very often, we'll have two equations and two unknown variables. There are several algebraic tactics that can be employed to solve the 'system'.

$$a + 4b = 2 \tag{1}$$
$$a - 4 = 4b \tag{2}$$