1. Lagrange Points

1.1 Preliminaries

Any conservative force can be related to potential energy by the following (in 1-D):

$$F(x) = -\frac{dU}{dx} \tag{1}$$

or in 3-D:

$$\mathbf{F} = -\nabla U \tag{2}$$

For example, mass on a spring:

$$F_x = -kx \tag{3}$$

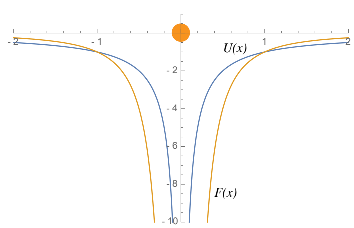
$$U_{\text{spring}} = \frac{1}{2}kx^2 \tag{4}$$

Or, for gravity near the surface of the Earth:

$$F_y = -mg \tag{5}$$

$$U_{\text{grav}} = mgy \tag{6}$$

1.2 Gravitational Potential



Potential Energy and Force due to a single gravitational source.

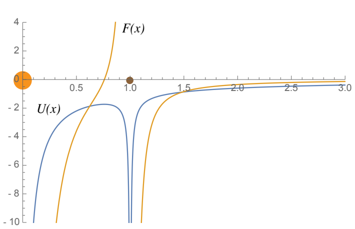
And, if we consider any distance away from the Earth (or central body), we would have:

$$F(r) = -G\frac{M_E m}{r^2} \tag{7}$$

$$U(r) = -G\frac{M_E m}{r} \tag{8}$$

Binary Systems

Potential Landscape



Adding another source makes a more complicated landscape

Now, if we have 2 source bodies, the functions have more features. Now, we ask what happens to a 3rd body in this landscape.

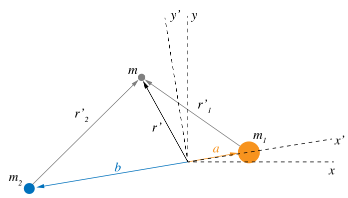
Binary Systems

1.3 Three Bodies

A different type of problem

Restricted 3 body

- m_1 much greater than m_2
- Circular Orbits
- m_3 is negligible in mass



Coordinates for a restricted 3 body system

Rotating Reference Frame

Forces in Rotating Reference Frame

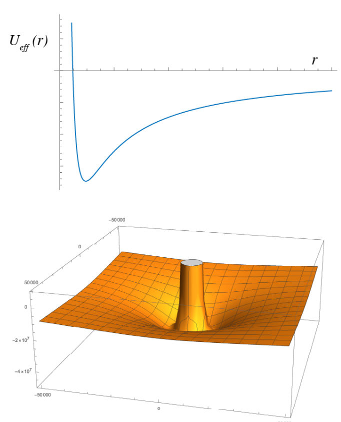
$$\mathbf{F}_{\text{rot}} = \mathbf{F}_{\text{in}} \underbrace{-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})_{\text{rot}}}_{\text{centrifugal}} \underbrace{-2m(\boldsymbol{\omega} \times \mathbf{v}_{\text{rot}})_{\text{rot}}}_{\text{Coriolis}} \underbrace{-m(\dot{\boldsymbol{\omega}} \times \mathbf{r})_{\text{rot}}}_{\text{Euler}} \tag{9}$$

For the case of gravity:

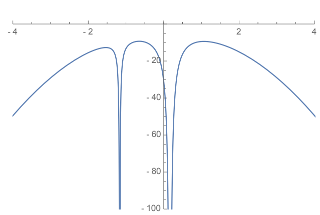
$$U_G(r) = -\frac{Gm_1m_2}{r} \tag{10}$$

and so our effective potential is:

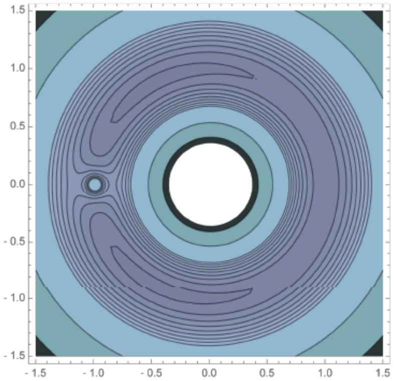
$$U_{\text{eff}} = \frac{l^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \tag{11}$$



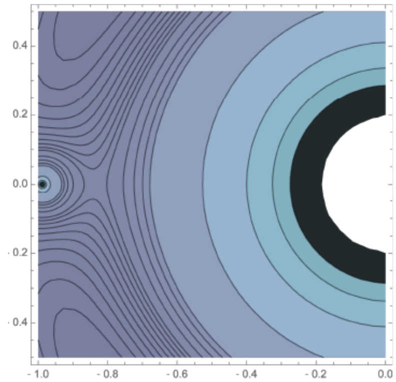
Effective Potential Gravity



The effective potential

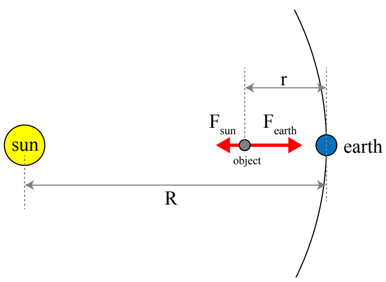


contour plot



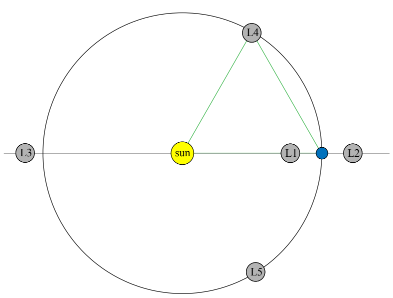
contour plot

2. Lagrange Points



The basic FBD and L1 point

The basic free body diagram showing the forces on a small body near two larger ones.



All 5 Lagrange points

There are five Lagrange points. Some are interesting for technology applications.

The sum of forces acting on the earth is equal to its mass time the acceleration, which for an object moving in a circular orbit is centripetal and equal to v^2/R:

$$\Sigma F_{\text{on Earth}} = \frac{GMm_E}{R^2} = \frac{m_E v^2}{R}$$

Thus we can say:

$$\frac{GM}{R} = v^2$$

but, based on the relation between speed and period, T:

$$v = \frac{2\pi R}{T}$$

or

$$v^2 = \frac{4\pi^2 R^2}{T^2}$$

So we can rewrite as:

$$\frac{GM}{R} = \frac{4\pi^2 R^2}{T^2}$$

Rearranging yields Kepler's Third Law

$$\frac{GM}{R^3} = \frac{4\pi^2}{T^2} \tag{12}$$

Now consider the sum of forces on the satellite:

$$\Sigma F_{\text{on Sat}} = \frac{GMm_{\text{sat}}}{(R-r)^2} - \frac{Gm_E m_{\text{sat}}}{r^2} = \frac{m_{\text{sat}} v_{\text{sat}}^2}{(R-r)}$$

This we can simplify to, using Kepler's Third law for the satellite:

$$\frac{GM}{R-r} - \frac{Gm_E(r-R)}{r^2} = v_{\text{sat}}^2 = \frac{4\pi^2 (R-r)^2}{T_{\text{sat}}^2}$$

or

$$\frac{GM}{(R-r)^3} - \frac{Gm_E}{r^2(R-r)} = \frac{4\pi^2}{T_{\text{sat}}^2} \tag{13}$$

Lastly, since we want their periods to be the same: $T_{\text{sat}} = T$

$$\frac{GM}{(R-r)^3} - \frac{Gm_E}{r^2(R-r)} = \frac{GM}{R^3} \tag{14}$$

If we put in the masses of the two major bodies, the Earth and Sun for example, we can calculate the value of r in terms of R. For our earth-sun system, the L1 point would be located at

$$r = 0.009969 \text{ AU}$$

or about 1/100 of the earth-sun distance. Changing signs in the sums of forces could allow to calculate the L2 and L3 positions as well.

L1 simulator